Panel 1

Last Time:

Alternative def's of derivative:

$$f'(x) = \lim_{t \to x} \frac{f(x) - f(t)}{x - t} \qquad \frac{d}{dx} f(x) \text{ or } \frac{df}{dx}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiable vs not differentiable

Not differentiable = sharp corners, eg $|x|$

or not continuous

Differentiable $\Rightarrow$ Continuity

Not conversely e.g. $|x|$

Panel 2

Differentiation Rules (1)

(1) $f(x) = x^n \Rightarrow f'(x) = n x^{n-1}$

Ex: $f(x) = x^5 \Rightarrow \frac{df}{dx} = 5x^4$

$f(x) = \frac{1}{x^4} = x^{-4}$, $f'(x) = -4x^{-5}$

$f(x) = \sqrt[3]{x^2} = x^{2/3}$, $\frac{df}{dx} = \frac{2}{3} x^{-1/3}$

Panel 3

Differentiation Rules (2)

$$\frac{d}{dx}\, c\cdot g(x) = c\cdot\frac{d}{dx}\, g(x)$$

$$\frac{d}{dx}\, c = 0$$

$f(x) = 5x^2 \Rightarrow f'(x) = 5\cdot 2x = 10x$

$f(x) = -\frac{3}{x^2} = -3x^{-2}$ $\quad f'(x) = -3(-2)\, x^{-3} = \frac{6}{x^3}$

$f(x) = \frac{5}{9\sqrt{x^3}} = \frac{5}{9}\, x^{-3/2}$ $\quad f'(x) = \frac{5}{9}\left(-\frac{3}{2}\right) x^{-5/2} = -\frac{5}{6}\, x^{-5/2}$

$f(x) = \pi^2$ $\quad f'(x) = 0$

Panel 4

Differentiation Rules (3)

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

Ex: $f(x) = x^2 + x^5 - \frac{1}{x} \Rightarrow f'(x) = 2x + 5x^4 + x^{-2}$

$f(x) = 3x^2(1-2x)$ $\quad f'(x) = \cancel{6x(-2)}$

$= 3x^2 - 6x^3$ $\quad \Rightarrow f'(x) = 6x - 18x^2$

$f(x) = \frac{5x^2 - 9\sqrt{x} + 3}{x} = \frac{5x^2}{x} - \frac{9\sqrt{x}}{x} + \frac{3}{x} = 5x - 9x^{-1/2} + 3x^{-1}$

$f'(x) = 5 - 9(-\tfrac{1}{2})x^{-3/2} + 3(-1)x^{-2}$

Panel 5

Ex: Find points for $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

I.e. $\underbrace{\text{tangent line has slope } 0}_{\text{derivative is zero.}}$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0 \Rightarrow x = 0$, $x = \sqrt{3}$, $x = -\sqrt{3}$ — critical points: points where deriv. is zero.

Panel 6

Ex: Position of particle is $f(t) = t^3 - 6t^2 + 9t$

a) velocity at time t

b) when is particle at rest?

c) when is particle moving forward?

d) acceleration at time t

d) Is deriv. of velocity ?? $\Rightarrow a(t) = 6t - 12$

a) $v(t)$ is deriv. of $d(t)$ $\Rightarrow$ $v(t) = 3t^2 - 12t + 9$

b) $v(t) = 0 = 3(t^2 - 4t + 3) = 3(t-3)(t-1) = 0$

$\Rightarrow t = 3, 1$

c) If $v(t) > 0 \Rightarrow 3(t-3)(t-1) > 0 \Rightarrow x < 1$ and $x > 3$

$+ \;|_1\; - \;|_3\; +$ sign of velocity

Panel 7

Is particle moving forward for x > 1, x < 3

$v(t) = 3t^2 - 12t + 9$

$= 3(t-3)(t-1)$

No speed: $t = 1, 3$

Before $t=1$: speed positive

between $1 < t < 3$: speed is negative (try $t = 2$)

$t > 3$: positive

Moving forward if $v > 0$:

Review: $x^2 + 5x + 6 > 0$

(1) $x^2 + 5x + 6 = 0$ $(x+2)(x+3) = 0$

(2) Take test point between

$+ \quad - \quad +$

$-4 \quad -3 \quad -2.5 \quad -2 \quad 0$

$x < -3$ or $x > -2$

Panel 8

Differentiation Rules (4) Product Rule

$$\frac{d}{dx} f(x) \cdot g(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$= \lim \frac{f(t)g(t) - f(x)g(x)}{t-x} =$$

$$= \lim \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t-x} =$$

$$= \lim_{t \to x} g(t) \cdot \boxed{\frac{f(t) - f(x)}{t-x}} + f(x) \boxed{\frac{g(t) - g(x)}{t-x}}$$

$$= g(x) \cdot f'(x) + f(x)\, g'(x)$$

Panel 9

Ex: $f(x) = x^2(2x-3) = 2x^3 - 3x^2$

(1) Product Rule: $f'(x) = 2x(2x-3) + x^2(2)$

(2) $f'(x) = 6x^2 - 6x$

$f(x) = (x^2-2x)(3x^2+9x-1)$

$f'(x) = (2x-2)(3x^2+9x-1) + (x^2-2x)(6x+9)$

Panel 10

Ex: $f(x) = x^2 \sin(x)$

$f'(x) = 2x \cdot \sin(x) + x^2 \cos(x)$

Hint: $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} = \cos(x)$$

$\to 0$ $\to 1$

Panel 11

Note: $\frac{d}{dx}\sin(x) = \cos(x)$ ✓

$\frac{d}{dx}\cos(x) = -\sin(x)$ check!

$\sin^2(x) = \sin(x)\cdot\sin(x)$

Ex: $f(x) = 5\sin(x) + \sin^2(x) - \frac{\sin(x)}{\cos(x)}$ $\quad \frac{d}{dx}\sin^2(x) = \frac{d}{dx}(\sin(x)\cdot\sin(x)) =$

$= \cos(x)\sin(x) + \sin(x)\cos(x)$

$f'(x) = 5\cos(x) + 2\cos(x)\sin(x) \; ? \; = 2\cos(x)\sin(x)$

$\sin^2(x) = 1-\cos^2(x) \qquad \sin(x) = \sqrt{1-\cos^2(x)}$

Panel 12

What is $\frac{d}{dx}\cos(x)$? $= -\sin(x)$

Panel 13

Differentiation Rules (5)

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \cancel{\frac{f'(x)}{g'(x)}} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$h(x) = \frac{x^2+7x}{x} = \frac{2x^2+\cancel{7x} - x^2 - \cancel{7x}}{x^2} = \frac{x^2}{x^2} = 1$$

1) Quotient: $h'(x) = \dfrac{(2x+7)\cdot x - (x^2+7x)(1)}{x^2}$

2) $h(x) = \frac{x^2}{x} + \frac{7x}{x} = x + 7 \rightarrow h'(x) = 1$

Panel 14

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} =$$

$$= \sec^2(x)$$

$\frac{d}{dx}\sec(x)$ HW

$\frac{d}{dx}\cot(x)$ HW

$\frac{d}{dx}\csc(x)$ HW

Panel 15

Ex: $f(x) = \frac{x^2+7x}{x^2}$

Panel 16

Higher Order Derivatives

$f(x)$ is a function

$\Rightarrow f'(x)$ is a function

$\Rightarrow (f')'(x) = f''(x)$ is 2nd deriv. of f

$f'''(x) =$ 3rd deriv

$f^{(n)}(x)$ is n^{th} - derivative

10th derivative!

$f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

$h(x) = x \cdot g(x)$

$h'(x) =$ ~~$g(x) +$~~

$x \cdot$

Panel 17

$h(x) = x \sin(x)$

$h'(x) = 1 \cdot \sin(x) + \underbrace{x \cos(x)}$

$h''(x) = \cos(x) + 1 \cdot \cos(x) - x \sin(x) =$

$= \underline{2\cos(x)} - x\, \underline{\sin(x)}$

$h'''(x) = -2\sin(x) - (1 \cdot \sin(x) + x\cos(x)) =$

$-2\sin(x) - (\sin(x) + x\cos(x)) =$

$= -3\sin(x) - x\cos(x)$

Panel 18

Note: Exam 1 on Monday!

Ex: $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \sin(x) & \text{if } x \geq 0 \end{cases}$

Is f differentiable at 0?

HW today, post Review exam on Tue. night.

Wed review, M Exam 1