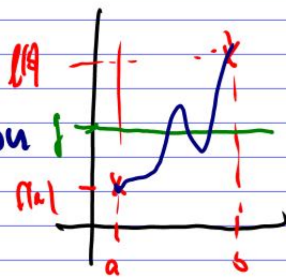
Panel 1

Last Time

Continuity: graph has no holes or gaps

$\lim_{x \to a} f(x) = f(a)$

IVT: If f is cont. on $[a,b]$ and y a number between $f(a)$ and $f(b)$. Then there is at least one $x \in (a,b)$ such that $f(x) = y$



IVT Corollary: If f is cont. on $[a,b]$, $f(a) \cdot f(b) < 0$ then there is $x \in (a,b)$ with $f(x) = 0$

Panel 2

$$f(x) = \begin{cases} \dfrac{3-\sqrt{x}}{9-x} & x \neq 9 \quad f(9) \text{ undefined} \\ \cancel{2}\ 1/6 & x = 9 \end{cases}$$

$\lim_{x \to 9} f(x) = f(9) = c$

$$\lim_{x \to 9} \frac{3-\sqrt{x}}{9-x} \cdot \frac{(3+\sqrt{x})}{(3+\sqrt{x})} = \lim_{x \to 9} \frac{\cancel{9-x}}{\cancel{(9-x)}(3+\sqrt{x})} = 1/6$$

$$f(x) = \begin{cases} \dfrac{x^3+64}{x+4} & x \neq -4 \\ 16 & x = 4 \end{cases} \qquad \frac{\cancel{(x+4)}(x^2-4x+16)}{\cancel{(x+4)}}$$

Panel 3

$x^4 + x - 3 = 0$ on $[1, 2]$

Let $f(x) = x^4 + x - 3$

f is a polyn. so continuous

$f(1) = -1$, $f(2) = 15$

Thus, by IVT, there is $c \in (1,2)$ with $f(c) = 0$

i.e. $c^4 + c - 3 = 0$

$f(1.5) = 3.56 \Rightarrow$ zero in $(1, 1.5)$

$= -4.5 \Rightarrow$ zero is $(1.5, 2)$

Panel 4

$c^2 = 2 \Rightarrow c = \sqrt{2}$?!

$f(x) = x^2 - 2$ Want to solve: $f(x) = x^2 - 2 = 0$

$x \in [0, 4]$ $f(0) < 0$, $f(4) > 0$

$\Rightarrow$ by IVT: $f(c) = 0$

Panel 5

The Story so far....

Limits:

- intuitive
- formal
- practical ways: $\frac{\#}{0}$, $\frac{0}{\#}$, $0/0$ more work
- at infinity
- graphically
- left/right
- Squeezing thm

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

Continuity

- intuitive
- formal
- Int. Value thm

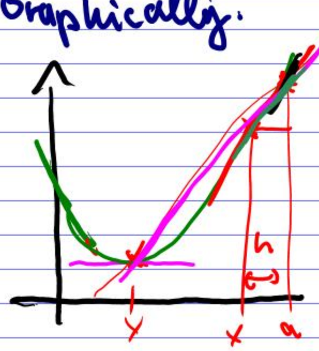
removable, jump, essential disc.

Panel 6

Next topic: Derivatives

Latin for "to touch"

Graphically: Want to find tangent line! (slope of)



slope is $\frac{\text{rise}}{\text{run}} = \frac{f(x)-f(a)}{x-a}$

future: $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is slope of tangent line!

awkward limit: $\frac{0}{0}$ no matter what f

Panel 7

Next topic: Derivatives

Speed (inst. velocity): Say the distance a particle has travelled at time $t=c$ is $d(t)=6t^2+3t$

What is the speed when time is $t=2$?

Speed = $\frac{\text{dist}}{\text{time}}$

Take $\frac{d(1)-d(2)}{1-2}$ avg. speed between 1 and 2

$\lim_{x\to 2} \frac{d(x)-d(2)}{x-2}$ speed at $x=2$

Panel 8

Def. of Derivative: Take $f(x)$ to be some function. Then the derivative of $f(x)$ at $x=a$ is defined as

$$\overset{\text{prime}}{f'(a)} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Derivative, or

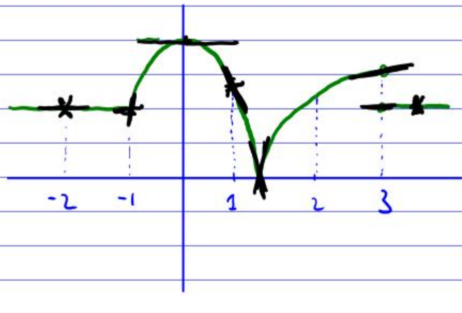
slope of tangent

speed (instantaneous)

(inst.) rate of change

Panel 9

Ex: Consider the following graph of a function f and find the indicated derivatives (pos, neg, zero)



$f'(0)$ zero

$f'(-2)$ zero

$f'(1)$ neg.

$f'(2)$ pos

$f'(3)$ undef'd

Where is f not continuous? $x=3$ ✓

Where is f not differentiable? $x=3$, $x=1.4$, $x=-1$

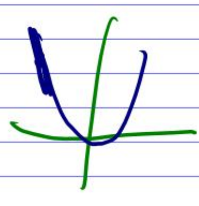
Continuity: no holes or gaps

Derivative: no kinks, cusps, or corners or sharp edges

Panel 10

Examples of Derivatives:

If $f(x) = x^2$, find $f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} =$



$$= \lim_{x \to 2} \frac{x^2 - 4}{x-2} =$$

$$= \lim_{x \to 2} \frac{(x+2)(\cancel{x-2})}{\cancel{x-2}} = 4$$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x \to 3} \frac{x^2 - 9}{x-3} = 6$$

$f'(\cancel{x}) =$

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t-x} = \lim_{t \to x} \frac{t^2 - x^2}{t-x} =$$

$$= \lim_{t \to x} \frac{(t+x)\cancel{(t-x)}}{\cancel{t-x}} = \underline{2x}$$

Panel 11

Def: $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

$t - x = h$

$t = x + h$

Alternate form:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ ⊛

$f(x) = x^2$, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \underline{\underline{2x}}.$$

Panel 12

Ex: $f(x) = x^3$, find $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} =$

$$= \lim_{t \to x} \frac{t^3 - x^3}{t - x} =$$

$$= \lim_{t \to x} \frac{\cancel{(t-x)}(t^2 + tx + x^2)}{\cancel{t-x}} = 3x^2$$

Ex: If $f(x) = \frac{1}{x}$, find $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} =$

$$= \lim_{t \to x} \frac{1/t - 1/x}{t - x} = \lim_{t \to x} \frac{\frac{x - t}{tx}}{t - x} =$$

$$= \lim_{t \to x} \frac{\cancel{x-t}}{tx} \cdot \frac{1}{\cancel{t-x}} = \lim_{t \to x} -\frac{1}{tx} = \underline{\underline{-\frac{1}{x^2}}}$$

Panel 13

$f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h}\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} =$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+\cancel{h})-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \underline{\frac{1}{2\sqrt{x}}}$$

Panel 14

__Theorem:__ If $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$. The converse is

X Test

Panel 15

Short cuts for differentiation:

$f(x) = x^2 \Rightarrow f'(x) = 2x^1$

Guess:

$f(x) = x^4 \Rightarrow f'(x) = 4x^3$

$f(x) = x^5 \Rightarrow f'(x) = 5x^4$

$f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}$

$f(x) = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2}$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -\frac{1}{x^2} = -x^{-2}$ ✓

$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$ ✓

Panel 16

Power Rule:

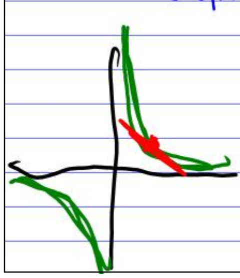
If $f(x) = x^p$ then $f'(x) = p\,x^{p-1}$

Ex: $f(x) = x^5 \Rightarrow f'(x) = 5x^4$

$f(x) = \frac{1}{x^2} = x^{-2} \quad f'(x) = -2x^{-3}$

$f(x) = \sqrt[3]{x} = x^{1/3} \quad f'(x) = \frac{1}{3}x^{-2/3}$

Slope of tangent line to $f(x) = 1/x^3$ at $x = 1$



$f(x) = 1/x^3 = x^{-3} \Rightarrow f'(x) = -3x^{-4}$

$\Rightarrow f'(1) = -3$

Panel 17

Ex: $f(x) = x^2 + x^7 \Rightarrow f'(x) = 2x + 7x^6$

Add / Subtract Thm $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

Ex: $f(x) = 3x^4$ $\quad (x^4 + x^4 + x^4)$

$f'(x) = 12x^3 \quad (4x^3 + 4x^3 + 4x^3)$

$= 3 \cdot 4x^3$

Const. theorem: $\left(c\, f(x)\right)' = c \cdot f'(x)$

Panel 18

Ex: $f(x) = 3x^2 - 7\sqrt{x} + \frac{2}{x^6} - \frac{1}{5\sqrt[3]{x}}$.

$\Rightarrow f(x) = 3x^2 - 7x^{1/2} + 2x^{-6} - \frac{1}{5}x^{-1/3}$

$f'(x) = 3 \cdot 2x^1 - 7\frac{1}{2}x^{-1/2} + 2(-6)x^{-7} - \frac{1}{5}\left(-\frac{1}{3}\right)x^{-4/3}$

$= 6x - \frac{7}{2}x^{-1/2} - 12x^{-7} + \frac{1}{15}x^{-4/3}$

$= 6x - \frac{7}{2\sqrt{x}} - \frac{12}{x^7} + \frac{1}{15\sqrt[3]{x^4}}$ or

Quiz on Wed!