Panel 1

Last Time

Limits $\lim_{x \to a} f(x) = L$

1) Substitute + hope for the best

2) $\frac{0}{\#} = 0$

$\frac{\#}{0}$ = not defined

$\frac{0}{0}$ = more work

X $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$

x close to a, but less than a (on minus side)

x close to a, but greater than a (on plus side)

X limits graphically of piecewise def. funct.

X $\lim_{x \to \pm\infty} f(x)$

Panel 2

Algebraic limits at infinity: Shortcut Rules

$$\lim_{x \to \infty} \frac{x^2-7x+9}{8-2x^2} = -\frac{1}{2} \quad \text{(tie)}$$

$$\lim_{x \to \infty} \frac{x^3-7x+8}{5x^4-9x} = 0$$

$$\lim_{x \to \infty} \frac{5-x^4}{x^3-2} = \cancel{+}-\infty$$

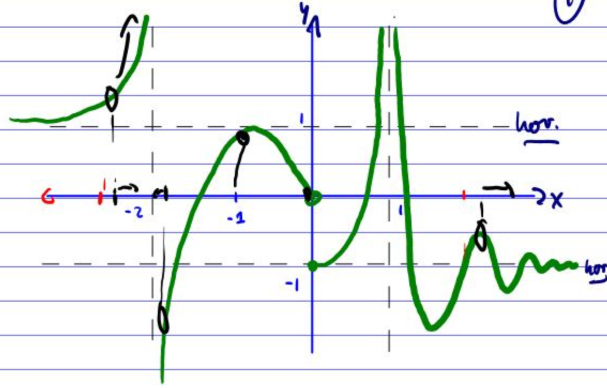
$$\lim_{x \to -\infty} \frac{5-x^4}{x^3-2} = +\infty$$

$$\lim_{x \to -\infty} \frac{5-x^4}{x^3-2} \sim -x^2 = -\infty$$

Panel 3

Limits Graphically

$x \to a^+$:



① a) $\lim\limits_{x \to \infty} f(x) = -1$

b) $\lim\limits_{x \to -\infty} f(x) = 1$

c) $\lim\limits_{x \to -1} f(x) \approx 1$

d) $\lim\limits_{x \to -1^-} f(x) = 1$

② a) $\lim\limits_{x \to -2^-} f(x) = \infty$ b) $\lim\limits_{x \to -2^+} f(x) = -\infty$ c) $\lim\limits_{x \to -2} f(x)$ d.n.e

③ a) $\lim\limits_{x \to 1^-} f(x) = \infty$ b) $\lim\limits_{x \to 1^+} f(x) = \infty$ c) $\lim\limits_{x \to 1} f(x) = \infty$

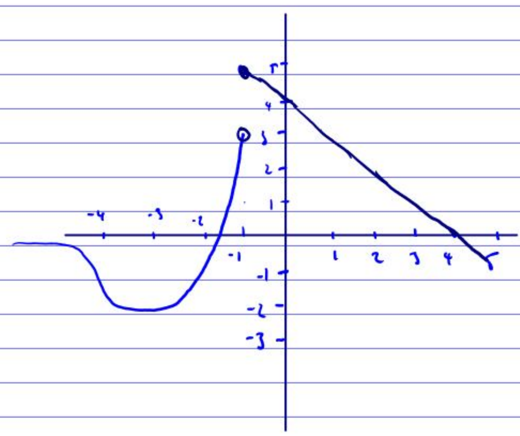
④ a) $\lim\limits_{x \to 0^-} f(x) = 0$ b) $\lim\limits_{x \to 0^+} f(x) = -1$ c) $\lim\limits_{x \to 0} f(x) =$ d.n.e.

Panel 4

Quiz #2

Name:

① The graph of a function is shown below. Find:



a) $\lim\limits_{x \to 4} f(x) =$

b) $\lim\limits_{x \to -1^-} f(x) =$

c) $\lim\limits_{x \to 0^+} f(x) =$

Panel 5

② Compute the following limits if they exist:

a) $\lim\limits_{x \to 0} \dfrac{x^2+2x-6}{3x+1}$

b) $\lim\limits_{x \to 3} \dfrac{x^2-9}{x^2-x-6}$

c) $\lim\limits_{x \to 1} f(x)$ for $f(x) = \begin{cases} 2x^2-1, & x \le 1 \\ 1-x, & x > 1 \end{cases}$

Panel 6

What would you pick for c if you had a choice?

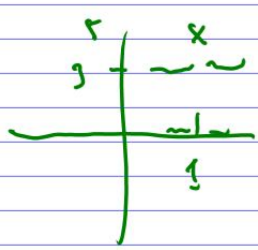
$$f(x) = \begin{cases} \dfrac{x^3-1}{x-1} & \text{if } x \neq 1 \\ c & \text{if } x = 1 \end{cases}$$

$f(1) = c \cancel{3}$?

Hint: $x^3 - 1 = (x-1)(x^2+x+1)$

pick $c=3$!

$$\lim_{x\to 1} f(x) = \lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 3$$



Panel 7

Limits formally

If x is close to a, $f(x)$ is close to L. No good! Not math!!

Def. of Limit: If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

then $\lim_{x \to a} f(x) = L$

Panel 8

Prove that $\lim_{x \to 1} 2x + 1 = 3$ You pick / I pick!

If for every number $\varepsilon > 0$ there is a $\delta > 0$ such that: if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$

0.002 $\delta = 0.001$ $\delta = \varepsilon/2$

Ex. You pick $\varepsilon = 0.25$ $= 0.01$ I must pick δ s.t. if $|x - 1| < \delta$ 0.01 then $|f(x) - L| < \varepsilon$, i.e. $|2x + 1 - 3| < 0.25$ 0.01

$|2x - 2| < 0.25$

$2|x - 1| < 0.25$ 0.01

$|x - 1| < \frac{0.25}{2} = \cancel{0.125}$ 0.005 0.01

I pick $\delta = \frac{0.25}{2} = \cancel{0.125}$ 0.005

Then: if $|x - 1| < \delta \Rightarrow |x - 1| < \cancel{0.125}$ 0.005

$\Rightarrow |2x + 1 - 3| < \cancel{0.25}$ 0.01

$\Rightarrow |f(x) - L| < \varepsilon$

Panel 9

Theorems about Limits

Suppose $\lim_{x\to a} f(x) = M$ and $\lim_{x\to a} g(x) = N$. Then:

$\lim_{x\to a} f(x) + g(x) = M+N$ eg $\lim_{x\to 1} 2x+5x^2 = 7$

$\lim_{x\to a} f(x) - g(x) = M-N$

$\lim_{x\to a} f(x)\cdot g(x) = M\cdot N$

$\lim_{x\to a} f(x)/g(x) = M/N$ if $N\neq 0$

Ex: $\lim_{x\to 0} \frac{2x-3}{(5+x^2)\sqrt{1-x}} = \frac{-3}{5\cdot 1} = -\frac{3}{5}$

Panel 10

The Squeezing Theorem

If $g(x) \le f(x) \le h(x)$ and if $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$

then $\lim_{x\to a} f(x) = L$

Trouble: $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ undef

Ex: $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by Squeezing Thm

Know: $-1 \le \sin\left(\frac{1}{x}\right) \le 1 \Rightarrow -x \le x\sin\left(\frac{1}{x}\right) \le x$

as $x\to 0$: $-x \to 0$, $x \to 0$

Panel 11

Misc. Limits memorize!

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \cos(x) \le \frac{\sin(x)}{x} \le 1$$

as $x \to 0$, $\to 1$ ✓

Hint: $\cos(x) \le \frac{\sin(x)}{x}$ and $\frac{\sin(x)}{x} \le 1$ trust me...

$$0 = \lim_{x\to 0} \frac{\cos(x)-1}{x} \cdot \frac{(\cos(x)+1)}{(\cos(x)+1)} = \lim_{x\to 0} \frac{\cos^2(x)-1}{x(\cos(x)+1)} = \lim \frac{\sin^2(x)}{x(\cos(x)+1)} =$$

memorize

$$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{\cos(x)+1} = 0$$

$\frac{\sin(x)}{x} \to 1$, $\frac{\sin(x)}{\cos(x)+1} \to 0$

$$\lim_{x\to 0} \frac{|x|}{x}$$

Panel 12

① $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ (Squeezing Thm)

memorize

② $\lim_{x\to 0} \frac{\cos(x)}{x}$ undef

③ $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$ (see ① + trick)

④ $\lim_{x\to 0} \frac{\cos(x)+1}{x}$ undef

⑤ $\lim_{x\to 0} \frac{\sin(x)}{x+1} = 0$

Panel 13

$$\lim_{x\to 0} \frac{\sin(x)}{3x} = \lim \frac{1}{3} \cdot \frac{\sin(x)}{x} = \frac{1}{3}$$

$$\lim_{x\to 0} \frac{\sin(7x)}{x} = \lim \cancel{\frac{7\sin(x)}{x}} = \lim_{x\to 0} \frac{7\cdot\sin(7x)}{7x} =$$

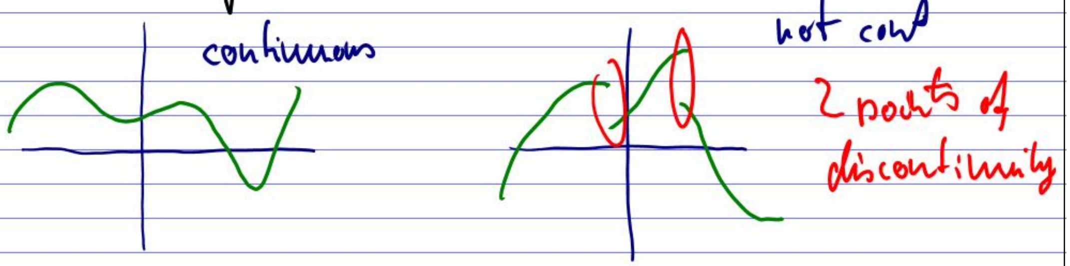
$$= \lim_{u\to 0} \frac{7\cdot\sin(u)}{u} = 7$$

with $u = 7x$.

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(5x)} = \lim_{x\to 0} \frac{4\,\underbrace{\frac{\sin(4x)}{4x}}_{\to 1}}{5\,\underbrace{\frac{\sin(5x)}{5x}}_{\to 1}} = \frac{4\cdot 1}{5\cdot 1} = \frac{4}{5}$$

Panel 14

Continuity



Continuity: draw graph without lifting pen!

Panel 15

Def. of Continuity: f is continuous at $x=a$ if:

(i) $f(a)$ exists, (ii) $\lim_{x \to a} f(x)$ exists

(iii) (i) = (ii)

Ex: $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(i) $f(2)$ undef

Continuous at $x=2$? Nope!!!

$= 3$

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ \cancel{7}\ 3 & \text{if } x = 2 \end{cases}$$

$= \lim_{x \to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} =$

Cont. at $x=2$? (i) $f(2) = \cancel{7}$ 3 (ii) $\lim_{x \to 2} f(x) = \lim_{x \to 2} \dfrac{x^2-x-2}{x-2}$

$\boxed{\text{Nope}}$ $\boxed{\text{Yup}}$

(iii) $\cancel{7}$ 3 $= 3$?

Panel 16

Continuity is easy if you see graph of $f(x)$.
Otherwise: continuity

Ex: $f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 2 \\ x + 1 & \text{if } x > 2 \end{cases}$ cont. at $x=2$???

(i) $f(2)$ = undef! No!

YES

(i) $f(2) = 3$

(ii) $\lim_{x \to 2} f(x) = 3$

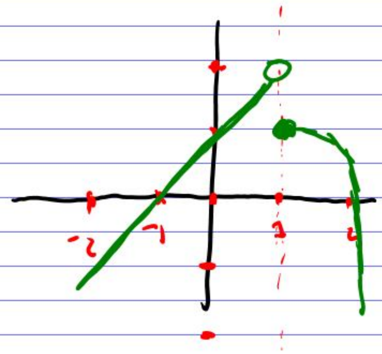
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 - 1 = 3$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + 1 = 3$

(i) = (ii) ✓

Panel 17

Example:



Not cont. for $x = 1$, cont. for all other x!

$g(x) = \dfrac{x^2-4}{x-2}$ Is cont. for $x \neq 2$, and NOT cont. at $x=2$

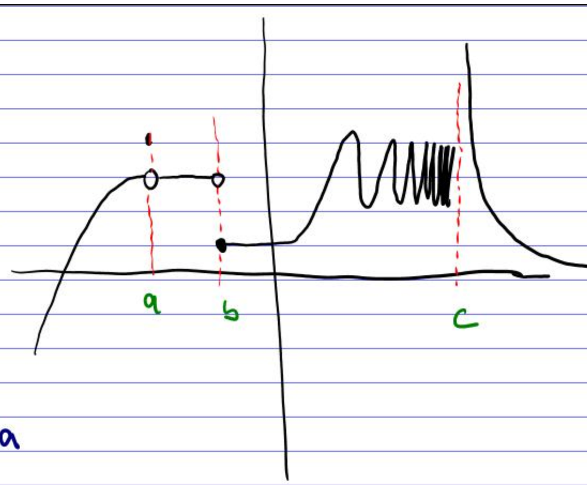
$$h(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ \cancel{3}\ 4 & \text{if } x = 2 \end{cases}$$

Cont. for all $x \neq 2$, ~~NOT~~ cont. at $x=2$?

Panel 18

Continuity Types

3 ways in which a function can fail to be cont.:



removable at $x=a$

jump at $x=b$

essential at $x=c$