Panel 1

Math 1501: Last time

Limits $\lim_{x \to a} f(x) = L$

"$f(x)$ gets close to L if x gets close to a, $x \neq a$"

To find limits:

- plug in, hope for the best
- $\frac{0}{\#} = 0$, $\frac{\#}{0} =$ d.n.e., n.a.n, $\frac{\infty}{\infty}$, $\frac{0}{0}$ more work — factoring, rationalizing, other tricks
- theorems

Panel 2

Compute the following limits:

a) $\lim_{x \to 2} \frac{x^2+1}{x+2} = \frac{5}{4}$

$\frac{x^2-4}{x-2} \neq x+2$

$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} x+2$

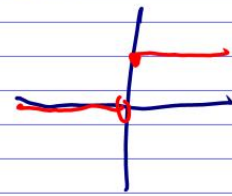
b) $\lim_{x \to 2} \frac{x^2-4}{x-2} \left(= \frac{0}{0}\right) = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = 4$

c) $\lim_{x \to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} = \frac{1}{6}$ ✓

Panel 3

Limits of Piecewise defined Functions:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



$\lim_{x \to 2} f(x) = 1$

limit d.n.e.

$\lim_{x \to -5} f(x) = 0$

if $\left(\begin{array}{c} x \text{ close to } 0 \\ x < 0 \end{array}\right)$ then answer is 0

$\lim_{x \to 0} f(x)$ not sure:

if $\left(\begin{array}{c} x \text{ close to } 0 \\ x > 0 \end{array}\right)$ then answer is 1

Panel 4

One-Sided Limits

$\lim_{x \to a^+} f(x)$ means x is close to a, but $x > a$

$\lim_{x \to a^-} f(x)$ means x is close to a, but $x < a$

Theorem: $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^+} f(x) = L$

and $\lim_{x \to a^-} f(x) = L$

Ex: $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ 2x + 1 & \text{if } x \geq 0 \end{cases}$ Find $\lim_{x \to 1} f(x) = 3$

$\lim_{x \to -2} f(x) = 3$

Panel 5

$$f(x) = \begin{cases} x^2-1 & \text{if } x<0 \\ 2x+1 & \text{if } x \geq 0 \end{cases}$$

$\lim_{x \to 0} f(x) = ?$ does not exist

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x+1) = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2-1) = -1$

Panel 6

Piecewise defined functions (again)

$$f(x) = \begin{cases} \frac{x^2-x-6}{x-3} & \text{if } x \neq 3 \\ 15 & \text{if } x=3 \end{cases} \qquad g(x) = \begin{cases} x^2-5 & \text{if } x \geq 0 \\ 2x-5 & \text{if } x<0 \end{cases}$$

a) $\lim_{x \to 0} f(x) = \frac{-6}{-3} = 2$

a) $\lim_{x \to 3} g(x)$

b) $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-x-6}{x-3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}} = 5$

b) $\lim_{x \to 0} g(x)$

$f(3) = 15$

Panel 7

$$g(x) = \begin{cases} x^2-5 & \text{if } x \geq 0 \\ 2x-5 & \text{if } x < 0 \end{cases}$$

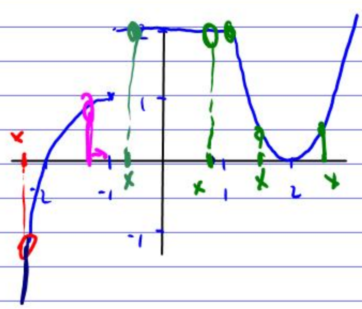
$g(1) = -4 \qquad g(-2) = -9 \qquad g(0) = -5$

a) $\lim_{x\to 3} g(x) = \lim_{x\to 3} x^2-5 = 9-5 = \underline{\underline{4}}$

b) $\lim_{x\to 0} g(x) = \underline{\underline{-5}} \Leftarrow \begin{cases} \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^2-5 = -5 \\ \lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} 2x-5 = -5 \end{cases}$

Panel 8

Limits Graphically:



$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to +1} f(x) = 2$

$\lim_{x\to -1^-} f(x) = 1$

$\lim_{x\to 0} f(x) = 2$

$\lim_{x\to 2} f(x) = 0$

$\lim_{x\to -1^+} f(x) = 2$

$\lim_{x\to 2^-} f(x) = 0$

$\Rightarrow \lim_{x\to -1} f(x) = \underline{dne}$

$\lim_{x\to 2^+} f(x) = 0$

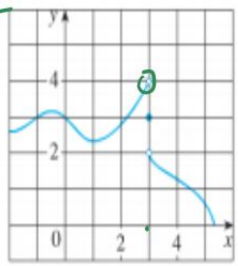
Panel 9

HW

4. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to 0} f(x)$ 3 (b) $\lim_{x\to 3^-} f(x)$ 4 (c) $\lim_{x\to 3^+} f(x)$ 2

(d) $\lim_{x\to 3} f(x)$ undef (e) $f(3)$ = 3



Panel 10

11. $\lim_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2}$

$x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
$1.9, 1.95, 1.99, 1.995, 1.999$

x	x
2.5	1.9
2.1	1.95
2.05	1.99
2.01	
0.666	0.666

$$\lim_{x\to 2} \frac{x^2-2x}{x^2-x-2} = \lim_{x\to 2} \frac{x(\cancel{x-2})}{(\cancel{x-2})(x+1)} = \frac{2}{3}$$

$$x = 2.001, \quad f(x) = \frac{0.002001}{0.003001} = \underline{0.66677...}$$

Panel 11

5. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x-1)^2 & \text{if } x \geqslant 1 \end{cases}$$

$\lim_{x \to -1^-} (2-x) = 3$

$\lim_{x \to -1} f(x) = \underline{\underline{\text{dne}}}$

$\lim_{x \to -1^+} x = -1$

$\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to +1} f(x) = \underline{\underline{\text{d.n.e}}}$

$\lim_{x \to 1^-} f(x) = 1$

Panel 12

In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

$\lim_{v \to c^-} L = \lim_{v \to c^-} L_0\sqrt{1 - v^2/c^2} = 0$

as $v \geq c$ is impossible (Einstein)

3:18

Panel 13

Limits at infinity

$\lim_{x\to\infty} f(x)$, i.e. what happens if x gets larger and larger...

$\lim_{x\to\infty} 2x^2 = \infty$

$\lim_{x\to+\infty} \frac{1}{-4x^3} = -0 = 0 \quad \left(\frac{1}{-\infty}\right)$

$\lim_{x\to\infty} \frac{1}{5x^3} = 0 \quad \left(= \frac{1}{\infty}\right)$

$\lim_{x\to\infty} \frac{3x^2+2x+1}{5x^3-7x+9} = 0$

think of it as a race!

Panel 14

$$\lim_{x\to\infty} \frac{2x^2-7x+1}{x^4-x^2+x-1} = \lim_{x\to\infty} \frac{x^2\left(2-\frac{7}{x}+\frac{1}{x^2}\right)}{x^4\left(1-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}\right)}$$

$$= \lim_{x\to\infty} \frac{2}{x^2} = 0$$

Shortcut:

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ \frac{\#}{\#} & \text{if } \deg(p) = \deg(q) \\ \pm\infty & \text{if } \deg(p) > \deg(q) \end{cases}$$

Panel 15

$$\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}=\begin{cases}0 & \text{if } \deg(p)<\deg(q)\\ \dfrac{\#}{\#} & \text{if } \deg(p)=\deg(q)\\ \pm\infty & \text{if } \deg(p)>\deg(q)\end{cases}$$

Ex (1) $\lim\limits_{x\to\infty}\dfrac{x^2-7}{5-x^4}=0$

$$\lim_{x\to-\infty}\frac{3x^3-8}{x^2+\cancel{x+1}}=\lim_{x\to-\infty}3x=-\infty$$

(2) $\lim\limits_{x\to-\infty}\dfrac{3x^2-2x+1}{-4x^2+x+7}=\dfrac{3}{-4}$

$$\lim_{x\to-\infty}\frac{4x^5-\cancel{8x}}{3x-2x+9}=$$

$$\lim_{x\to\infty}\frac{3x^3-7}{x^2+x+1}=\infty$$

$$\lim_{x\to-\infty}\frac{4}{3}x^4=+\infty$$

$-\infty$