

HIGHER DERIVATIVES

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f . Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

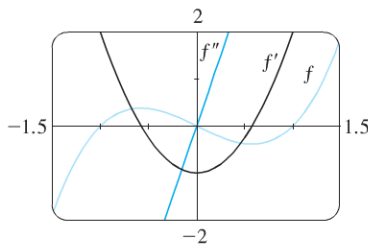


FIGURE 10



In Module 2.2 you can see how changing the coefficients of a polynomial f affects the appearance of the graphs of f , f' , and f'' .

The graphs of f , f' , f'' are shown in Figure 10.

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = \frac{d^3 y}{dx^3}$$

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the n th derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x . So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x ,

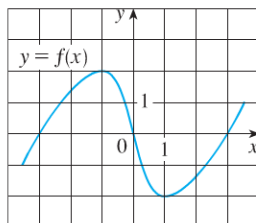
$$f^{(4)}(x) = 0$$

We have seen that one application of second derivatives occurs in analyzing the motion of objects using acceleration. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f . In Section 8.7 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

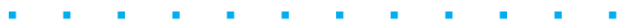
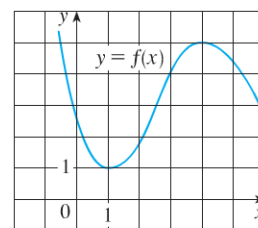
2.2 EXERCISES

1–2 ■ Use the given graph to estimate the value of each derivative. Then sketch the graph of f' .

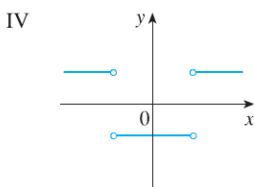
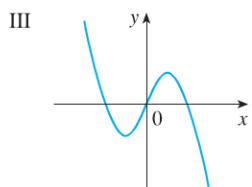
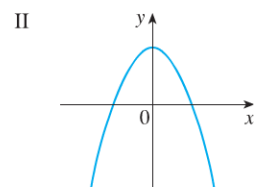
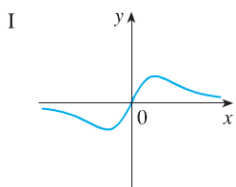
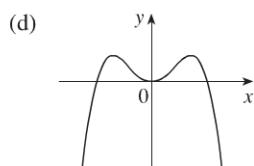
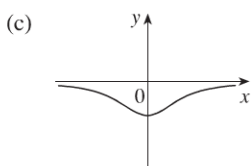
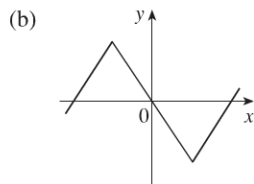
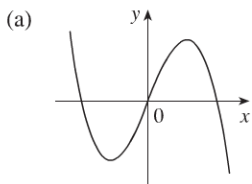
1. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$
 (d) $f'(0)$ (e) $f'(1)$ (f) $f'(2)$
 (g) $f'(3)$



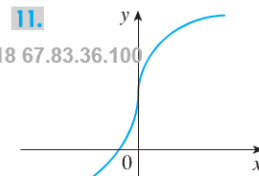
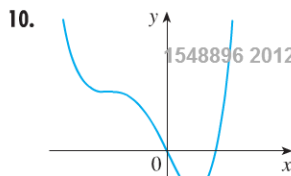
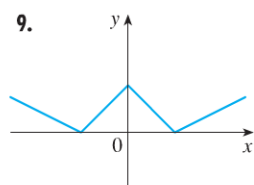
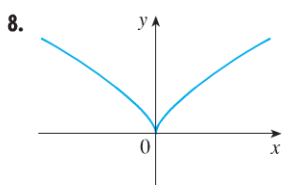
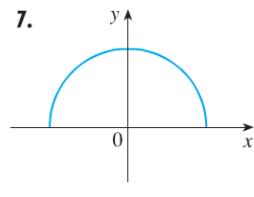
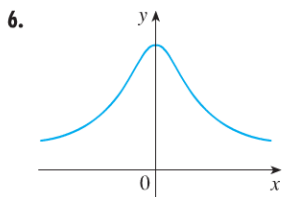
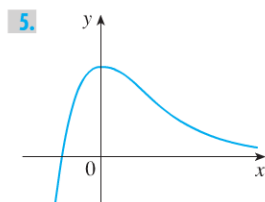
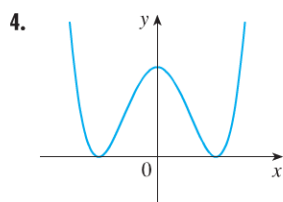
2. (a) $f'(0)$ (b) $f'(1)$
 (c) $f'(2)$ (d) $f'(3)$
 (e) $f'(4)$ (f) $f'(5)$



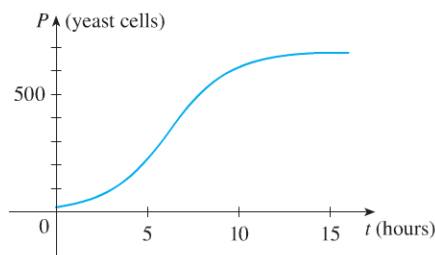
3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



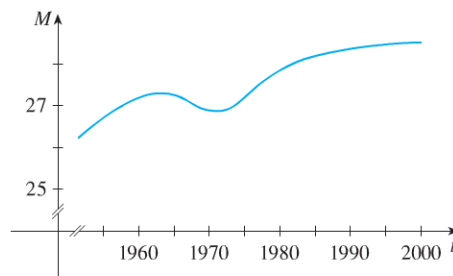
4–11 ■ Trace or copy the graph of the given function f . (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.



12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. The graph shows how the average age of first marriage of Japanese men has varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



14. Make a careful sketch of the graph of the sine function and below it sketch the graph of its derivative in the same manner as in Exercises 4–11. Can you guess what the derivative of the sine function is from its graph?



15. Let $f(x) = x^2$.

- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f .
- Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
- Use the results from parts (a) and (b) to guess a formula for $f'(x)$.

(d) Use the definition of a derivative to prove that your guess in part (c) is correct.

16. Let $f(x) = x^3$.
- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f .
 - Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
 - Use the values from parts (a) and (b) to graph f' .
 - Guess a formula for $f'(x)$.
 - Use the definition of a derivative to prove that your guess in part (d) is correct.

17–23 ■ Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

17. $f(x) = \frac{1}{2}x - \frac{1}{3}$ 18. $f(x) = 1.5x^2 - x + 3.7$

19. $f(x) = x^3 - 3x + 5$ 20. $f(x) = x + \sqrt{x}$

21. $g(x) = \sqrt{1 + 2x}$ 22. $f(x) = \frac{3 + x}{1 - 3x}$

23. $G(t) = \frac{4t}{t + 1}$

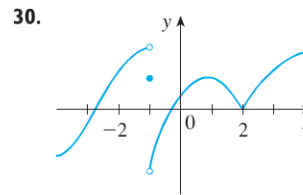
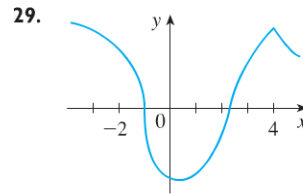
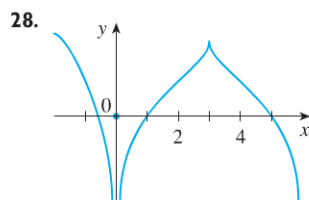
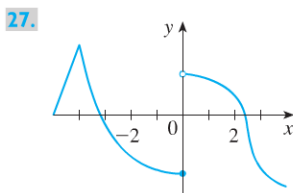
24. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.2.
 (b) Use the graph from part (a) to sketch the graph of f' .
 (c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f' ?

24. (d) Use a graphing device to graph f' and compare with your sketch in part (b).

25. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.
 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

26. (a) If $f(t) = t^2 - \sqrt{t}$, find $f'(t)$.
 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

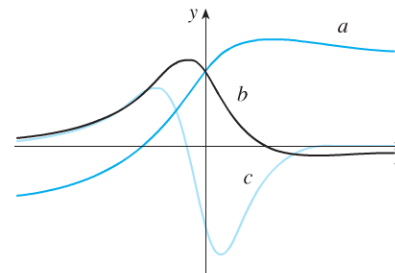
27–30 ■ The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



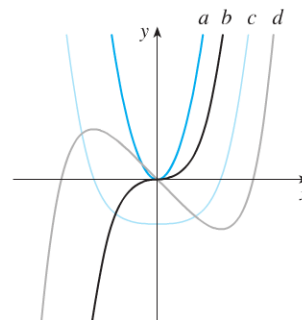
31. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f ?

32. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g .

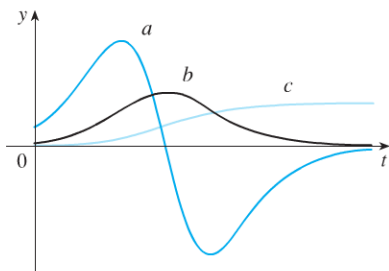
33. The figure shows the graphs of f , f' , and f'' . Identify each curve, and explain your choices.



34. The figure shows graphs of f , f' , f'' , and f''' . Identify each curve, and explain your choices.



35. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



- 36–37 ■ Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f , f' , and f'' on a common screen and check to see if your answers are reasonable.

36. $f(x) = 1/x$

37. $f(x) = 1 + 4x - x^2$

38. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f , f' , f'' , and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

39. Let $f(x) = \sqrt[3]{x}$.
 (a) If $a \neq 0$, use Equation 2.1.5 to find $f'(a)$.
 (b) Show that $f'(0)$ does not exist.

- (c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f . See Figure 8 in Section 1.2.)

40. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
 (b) If $a \neq 0$, find $g'(a)$.
 (c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
 (d) Illustrate part (c) by graphing $y = x^{2/3}$.



41. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.
42. Where is the greatest integer function $f(x) = \llbracket x \rrbracket$ not differentiable? Find a formula for f' and sketch its graph.
43. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x . Prove each of the following.
 (a) The derivative of an even function is an odd function.
 (b) The derivative of an odd function is an even function.
44. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.
 (a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
 (b) Describe how the rate of change of T with respect to t varies as t increases.
 (c) Sketch a graph of the derivative of T .
45. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x -axis. Calculate ϕ correct to the nearest degree.

2.3 BASIC DIFFERENTIATION FORMULAS

If it were always necessary to compute derivatives directly from the definition, as we did in the preceding section, such computations would be tedious and the evaluation of some limits would require ingenuity. Fortunately, several rules have been developed for finding derivatives without having to use the definition directly. These formulas greatly simplify the task of differentiation.

In this section we learn how to differentiate constant functions, power functions, polynomials, and the sine and cosine functions. Then we use this knowledge to compute rates of change.

Let's start with the simplest of all functions, the *constant function* $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

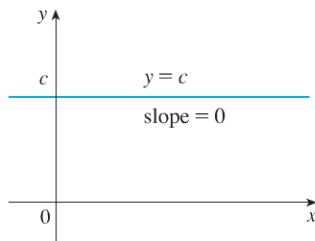


FIGURE 1
 The graph of $f(x) = c$ is the line $y = c$, so $f'(x) = 0$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} 0 = 0 \end{aligned}$$