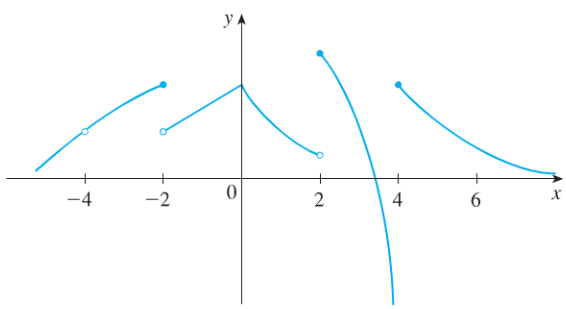
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

1.5 EXERCISES

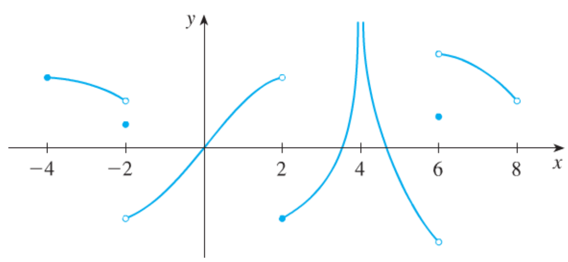
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x \to 3} [2f(x) - g(x)] = 4$, find $g(3)$.

10–11 ▪ Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

10. $f(x) = x^2 + \sqrt{7 - x}, \quad a = 4$

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. Use the definition of continuity and the properties of limits to show that the function $f(x) = x\sqrt{16 - x^2}$ is continuous on the interval $[-4, 4]$.

13–16 ▪ Explain why the function is discontinuous at $a = 1$. Sketch the graph of the function.

13. $f(x) = -\dfrac{1}{(x-1)^2}$

14. $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

15. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

16. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

17–22 ▪ Explain, using Theorems 4, 5, 6, and 8, why the function is continuous at every number in its domain. State the domain.

17. $F(x) = \dfrac{x}{x^2 + 5x + 6}$

18. $G(x) = \sqrt[3]{x}\,(1 + x^3)$

19. $R(x) = x^2 + \sqrt{2x - 1}$

20. $h(x) = \dfrac{\sin x}{x + 1}$

21. $F(x) = \sqrt{x}\,\sin x$

22. $F(x) = \sin(\cos(\sin x))$

23–24 ▪ Locate the discontinuities of the function and illustrate by graphing.

23. $y = \dfrac{1}{1 + \sin x}$

24. $y = \tan \sqrt{x}$

25–26 ▪ Use continuity to evaluate the limit.

25. $\lim_{x \to 4} \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$

26. $\lim_{x \to \pi} \sin(x + \sin x)$

27–28 ▪ Show that f is continuous on $(-\infty, \infty)$.

27. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geqslant 1 \end{cases}$

28. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

29. Find the numbers at which the function

$$f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

is discontinuous. At which of these points is f continuous from the right, from the left, or neither? Sketch the graph of f.

30. The gravitational force exerted by the Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\[2mm] \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of the Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

31. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

32. Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geqslant 4 \end{cases}$$

33. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}$, $a = -2$

(b) $f(x) = \dfrac{x - 7}{|x - 7|}$, $a = 7$

(c) $f(x) = \dfrac{x^3 + 64}{x + 4}$, $a = -4$

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$, $a = 9$

34. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

35. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

36. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

37–40 ▪ Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

37. $x^4 + x - 3 = 0$, $(1, 2)$

38. $\sqrt[3]{x} = 1 - x$, $(0, 1)$

39. $\cos x = x$, $(0, 1)$

40. $\tan x = 2x$, $(0, 1.4)$

41–42 ▪ (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

41. $\cos x = x^3$

42. $x^5 - x^2 + 2x + 3 = 0$

43–44 ▪ (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

43. $x^5 - x^2 - 4 = 0$

44. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

45. Is there a number that is exactly 1 more than its cube?