3–9 ▪ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to -2} (3x^4 + 2x^2 - x + 1)$

4. $\lim_{t\to -1} (t^2 + 1)^3(t + 3)^5$

5. $\lim_{x\to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

6. $\lim_{u\to -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x\to 1} \left(\frac{1 + 3x}{1 + 4x^2 + 3x^4}\right)^3$

8. $\lim_{x\to 0} \frac{\cos^4 x}{5 + 2x^3}$

9. $\lim_{\theta\to(\pi/2)} \theta \sin\theta$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} (x + 3)$$

is correct.

11–24 ▪ Evaluate the limit, if it exists.

11. $\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$

12. $\lim_{x\to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4}$

13. $\lim_{x\to 2} \frac{x^2 - x + 6}{x - 2}$

14. $\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim_{t\to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x\to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

17. $\lim_{h\to 0} \frac{(4 + h)^2 - 16}{h}$

18. $\lim_{h\to 0} \frac{\sqrt{1 + h} - 1}{h}$

19. $\lim_{x\to -2} \frac{x + 2}{x^3 + 8}$

20. $\lim_{x\to -1} \frac{x^2 + 2x + 1}{x^4 - 1}$

21. $\lim_{x\to 7} \frac{\sqrt{x + 2} - 3}{x - 7}$

22. $\lim_{h\to 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$

23. $\lim_{x\to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

24. $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t}\right)$

25. (a) Estimate the value of

$$\lim_{x\to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

26. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

27. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \cos 20\pi x = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

28. Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \sqrt{x^3 + x^2} \sin\frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

29. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x\to 4} f(x)$.

30. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

31. Prove that $\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0$.

32. Prove that $\lim_{x\to 0^+} \sqrt{x}\,[1 + \sin^2(2\pi/x)] = 0$.

33–36 ▪ Find the limit, if it exists. If the limit does not exist, explain why.

33. $\lim_{x\to 3} (2x + |x - 3|)$

34. $\lim_{x\to -6} \frac{2x + 12}{|x + 6|}$

35. $\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

36. $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

37. Let

$$g(x) = \begin{cases} -x & \text{if } x \leqslant -1 \\ 1 - x^2 & \text{if } -1 < x < 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.

(i) $\lim_{x\to 1^+} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $\lim_{x\to 0} g(x)$

(iv) $\lim_{x\to -1^-} g(x)$ (v) $\lim_{x\to -1^+} g(x)$ (vi) $\lim_{x\to -1} g(x)$

(b) Sketch the graph of g.

38. Let $F(x) = \frac{x^2 - 1}{|x - 1|}$.

(a) Find

(i) $\lim_{x\to 1^+} F(x)$ (ii) $\lim_{x\to 1^-} F(x)$

(b) Does $\lim_{x \to 1} F(x)$ exist?

(c) Sketch the graph of F.

39. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 8, evaluate

(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

40. Let $f(x) = x - [\![x]\!]$.

(a) Sketch the graph of f.

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} f(x)$ (ii) $\lim_{x \to n^+} f(x)$

(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

41. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

42. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

43–48 ▪ Find the limit.

43. $\lim_{x \to 0} \dfrac{\sin 3x}{x}$

44. $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 6x}$

45. $\lim_{t \to 0} \dfrac{\tan 6t}{\sin 2t}$

46. $\lim_{t \to 0} \dfrac{\sin^2 3t}{t^2}$

47. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

48. $\lim_{x \to 0} x \cot x$

49. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

50. If r is a rational function, use Exercise 49 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

51. To prove that sine has the Direct Substitution Property we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. If we let $h = x - a$, then $x = a + h$ and $x \to a \iff h \to 0$. So an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (1) to show that this is true.

52. Prove that cosine has the Direct Substitution Property.

53. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

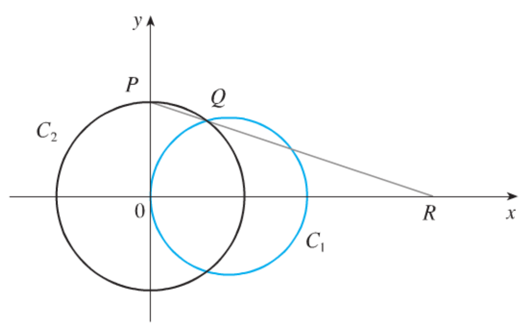
54. Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

55. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

56. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



1.5 CONTINUITY

We noticed in Section 1.4 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)