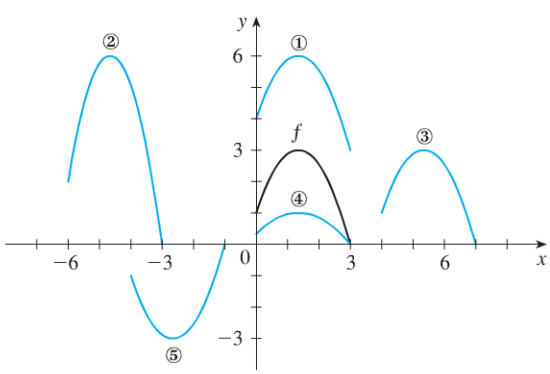
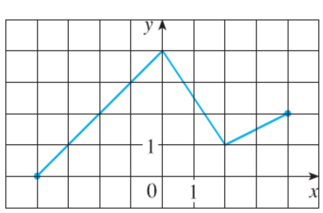
15. Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.
(a) Shift 3 units upward.
(b) Shift 3 units downward.
(c) Shift 3 units to the right.
(d) Shift 3 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 3.
(h) Shrink vertically by a factor of 3.

16. Explain how the following graphs are obtained from the graph of $y = f(x)$.
(a) $y = 5f(x)$
(b) $y = f(x - 5)$
(c) $y = -f(x)$
(d) $y = -5f(x)$
(e) $y = f(5x)$
(f) $y = 5f(x) - 3$

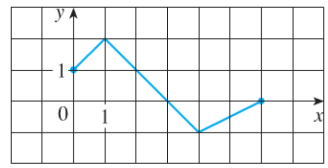
17. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
(a) $y = f(x - 4)$
(b) $y = f(x) + 3$
(c) $y = \frac{1}{3}f(x)$
(d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$



18. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x + 4)$ (b) $y = f(x) + 4$
(c) $y = 2f(x)$ (d) $y = -\frac{1}{2}f(x) + 3$



19. The graph of f is given. Use it to graph the following functions.
(a) $y = f(2x)$ (b) $y = f\left(\frac{1}{2}x\right)$
(c) $y = f(-x)$ (d) $y = -f(-x)$



20. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 18(a) to sketch the graph of $y = 2 \sin x$.
(b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 17(a) to sketch the graph of $y = 1 + \sqrt{x}$.

21–34 ■ Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations.

21. $y = -x^3$

22. $y = 1 - x^2$

23. $y = (x + 1)^2$

24. $y = x^2 - 4x + 3$

25. $y = 1 + 2 \cos x$

26. $y = 4 \sin 3x$

27. $y = \sin(x/2)$

28. $y = \dfrac{1}{x - 4}$

29. $y = \sqrt{x + 3}$

30. $y = (x + 2)^4 + 3$

31. $y = \frac{1}{2}(x^2 + 8x)$

32. $y = 1 + \sqrt[3]{x - 1}$

33. $y = \dfrac{2}{x + 1}$

34. $y = \dfrac{1}{4} \tan\left(x - \dfrac{\pi}{4}\right)$

35–36 ■ Find $f + g$, $f - g$, fg, and f/g and state their domains.

35. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

36. $f(x) = \sqrt{1 + x}$, $g(x) = \sqrt{1 - x}$

37–42 ■ Find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

37. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

38. $f(x) = 1 - x^3$, $g(x) = 1/x$

39. $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$

40. $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$

41. $f(x) = x + \dfrac{1}{x}$, $g(x) = \dfrac{x + 1}{x + 2}$

42. $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$

43–44 ▪ Find $f \circ g \circ h$.

43. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 2, \quad h(x) = x + 3$

44. $f(x) = \dfrac{2}{x + 1}, \quad g(x) = \cos x, \quad h(x) = \sqrt{x + 3}$

45–48 ▪ Express the function in the form $f \circ g$.

45. $F(x) = (x^2 + 1)^{10}$

46. $F(x) = \sin(\sqrt{x})$

47. $u(t) = \sqrt{\cos t}$

48. $u(t) = \dfrac{\tan t}{1 + \tan t}$

49–51 ▪ Express the function in the form $f \circ g \circ h$.

49. $H(x) = 1 - 3^{x^2}$

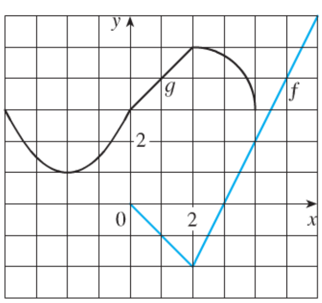
50. $H(x) = \sqrt[8]{2 + |x|}$

51. $H(x) = \sec^4(\sqrt{x})$

52. Use the table to evaluate each expression.
(a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
(d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

53. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.
(a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
(d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



54. A spherical balloon is being inflated and the radius of the balloon is increasing at a rate of 2 cm/s.
(a) Express the radius r of the balloon as a function of the time t (in seconds).
(b) If V is the volume of the balloon as a function of the radius, find $V \circ r$ and interpret it.

55. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
(a) Express the radius r of this circle as a function of the time t (in seconds).
(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

56. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t.
(b) Express the distance s between the plane and the radar station as a function of d.
(c) Use composition to express s as a function of t.

57. The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.
(a) Sketch the graph of the Heaviside function.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

58. The Heaviside function defined in Exercise 57 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.
(a) Sketch the graph of the ramp function $y = tH(t)$.
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 60$.
(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leqslant 32$.

59. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph?

60. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find $A \circ A$, $A \circ A \circ A$, and $A \circ A \circ A \circ A$. What do these compositions represent? Find a formula for the composition of n copies of A.

61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
(b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.

63. (a) Suppose f and g are even functions. What can you say about $f + g$ and fg?
(b) What if f and g are both odd?

64. Suppose f is even and g is odd. What can you say about fg?

65. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?

66. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

1.3 THE LIMIT OF A FUNCTION

Our aim in this section is to explore the meaning of the limit of a function. We begin by showing how the idea of a limit arises when we try to find the velocity of a falling ball.

V EXAMPLE 1 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} \\ &= \frac{s(5.1) - s(5)}{0.1} \\ &= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s} \end{aligned}$$

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

The table shows the results of similar calculations of the average velocity over successively smaller time periods. It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$ ■