Panel 1

Last Time

Optimization: ✓
Implicit Diff. ✓
Related Rates: ✓
Linearization: $f(x) \approx f'(c)(x-c)+f(c)$, x near c.

Differentials, Error Estimates, Relative Errors

$$dy = f'(x)\,dx$$

Ex: $f(x) = x^3 \Rightarrow dy = 3x^2\,dx$

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Panel 2

Ex: Suppose you compute the area of a square using its side length. What is the relative error in area if you measure x as $10\text{cm} \pm 0.5\text{cm}$?

$A = x^2 \qquad dA = 2x\,dx$

$= 2 \cdot 10 \cdot 0.5 = 20 \cdot 0.5 = \underline{10}$

Error in area is $\pm 10\text{cm}^2$, we want relative errors:

$$\frac{dx}{x} = \frac{0.5}{10} = 0.05 = 5\%$$

$$\frac{dA}{A} = \frac{10}{100} = 0.1 = 10\%$$

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Panel 3

Summary of Applications of Derivatives

1. Curve sketching
2. Optimization
3. Implicit diff
4. Related Rates
5. Linear Approx
6. Relative Error

applied applications

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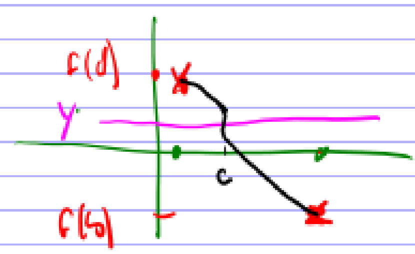
Panel 4

Final Application: Mean Value Theorem (MVT)

Recall IVT: Int. Value Thm: If y is between $f(a)$ and $f(b)$, then there is $c \in (a,b)$ s.t. $f(c) = y$, if f is cont.



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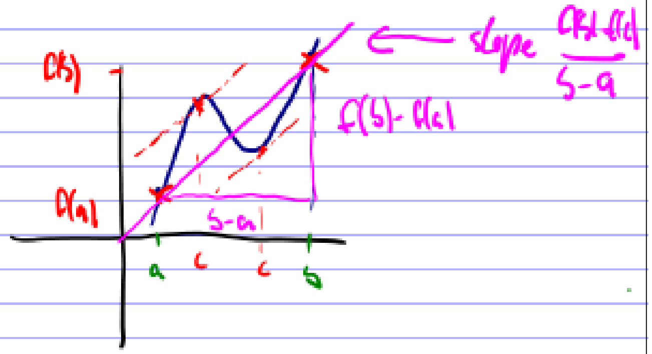
Panel 5

MVT:

If f is diff'ble on $[a,b]$, then there is $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$



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Panel 6

Ex: Consider $f(x) = x^3 - x$ on $[0,2]$. Find all values c that the MVT specifies.

MVT: $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{6}{2} = 3 = f'(c)$ $c \in (0,2)$

$= 3c^2 - 1$

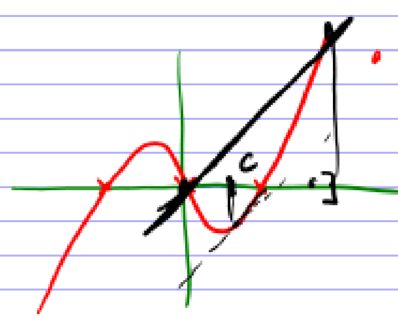
$f'(x) = 3x^2 - 1$

$\Rightarrow 3c^2 = 4 \Rightarrow c^2 = 4/3 \Rightarrow c = \cancel{\pm}\sqrt{4/3}$

$f(x) = x^3 - x =$

$= x(x^2-1) =$

$= x(x+1)(x-1)$



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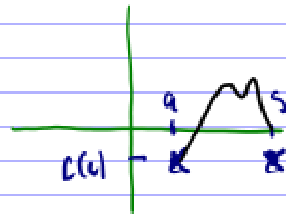
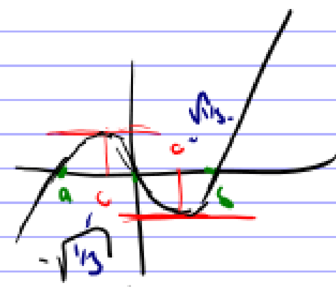
Panel 7

Rolle's Thm: f diff'ble on $[a,b]$ and $f(a)=f(b)$.
Then there is $c \in (a,b)$ s.t. $f'(c)=0$





Ex: $f(x)=x^3-x$ on $[-1,1]$. Find c s.t. $f'(c)=0$

$f(-1)=-1+1=0=f(1)$ $\quad f'=3x^2-1=0 \Rightarrow x=\pm\sqrt{1/3}$

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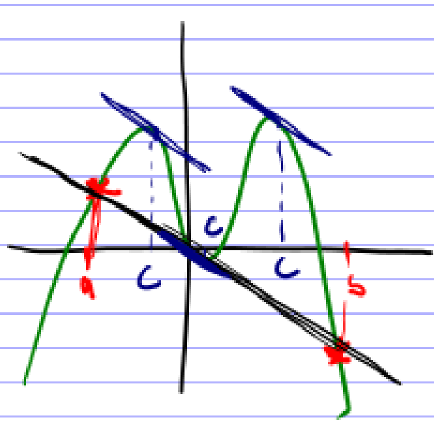
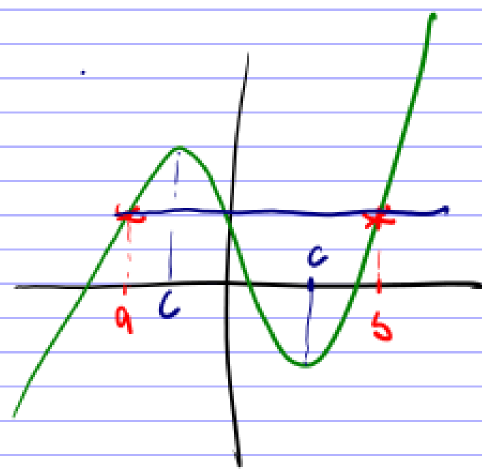
Panel 8

Quiz Question





MVT: 3 c's

Rolle

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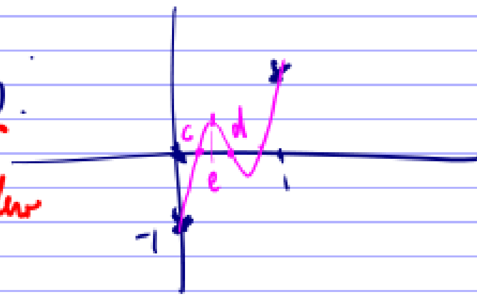
Panel 9

Ex: Show that $x^3+x-1=0$ has exactly one solution on $[0,1]$

$f(x)=x^3+x-1 \Rightarrow f(0)=-1,\ f(1)=1$

By IVT there is a number c where $f(c)=0$.



What if there was another d s.t. $f(d)=0$

$\Rightarrow$ by Rolle's theorem, there must be a e s.t. $f'(e)=0$
BUT $f'(x)=3x^2+1 \neq 0$ for any x. $\Rightarrow$ There can be no other d with $f(d)=0$

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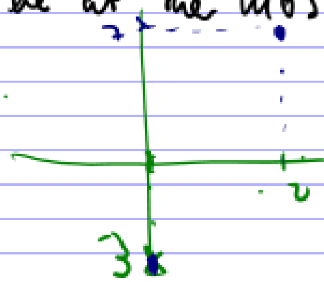
Panel 10

Ex: Suppose f is diff'ble $f(0)=-3$ and $f'(x)\leq 5$ on $[0,2]$.
How large can $f(2)$ be at the most?



MVT: $\frac{f(b)-f(a)}{b-a}=f'(c)$

$\frac{f(2)-f(0)}{2-0}=f'(c)\leq 5$

$f(2)-f(0)\leq 5\cdot 2=10 \Rightarrow f(2)\leq 10+f(0)=10-3=7$

Thm: Prove that if $f'(x)=0$ on $[a,b]$ then

$\Rightarrow$ MVT: $\frac{f(b)-f(a)}{b-a}=f'(c)=0 \Rightarrow f(b)-f(a)=0$

$\Rightarrow f(a)=f(b) \Rightarrow$ means f is constant!

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Panel 11

Inverse Functions and Derivatives

inverse of x^2 is $\sqrt{x}$, of x^3 it is $\sqrt[3]{x}$

of $3x$ is $\frac{1}{3}x$

Def: If f is a function, and g is another s.t. $f(g(x)) = x = g(f(x))$, then g and f are inverse and we write $g(x) = f^{-1}(x)$

Ex: Are $f(x) = 3x+2$ and $\frac{1}{3}x - 2$ inverse?

Check $f(g(x)) = f(\frac{1}{3}x-2) = 3(\frac{1}{3}x-2)+2 = x-4$

Nope!

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Panel 12

How to find inverses:

1. $y = f(x)$ solve for x
2. Flip x and y.

$f(x) = 3x+2$ → find inverse:

$y = 3x+2 \Rightarrow y-2 = 3x \Rightarrow \frac{1}{3}(y-2) = x$

$\Rightarrow f^{-1}(x) = \frac{1}{3}(x-2)$

check: $f(f^{-1}(x)) = f(\frac{1}{3}(x-2)) = 3(\frac{1}{3}(x-2))+2 = x-2+2 = x$

$f^{-1}(f(x)) = f^{-1}(3x+2) = \frac{1}{3}(3x+2-2) = x$ ✓

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Panel 13

Find inverse of $f(x) = \frac{x-1}{x+1}$

$y = \frac{x-1}{x+1}$

$y(x+1) = x-1$

$yx + y = x - 1$

$y + 1 = x - yx$

$y+1 = x(1-y)$

$\frac{y+1}{1-y} = x \Rightarrow y = \frac{x+1}{1-x}$

Check: $f^{-1}(x) = \frac{x+1}{1-x}$ is inverse of $f(x) = \frac{x-1}{x+1}$

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Panel 14

Theorem: If f is continuous and f has an inverse function then f^{-1} is continuous.

Theorem: If f is diff'ble and f has an inverse function then f^{-1} is diff'ble and

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Ex: $f(x) = x^2 \Rightarrow f^{-1}(x) = \sqrt{x}$

$f'(x) = 2x \Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{2 \cdot \sqrt{x}}$

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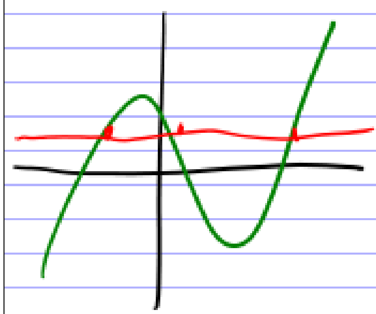
Panel 15

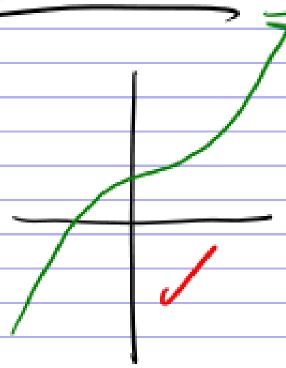
Ex: Find $(f^{-1})'(1)$ for $f(x) = 2x+\cos(x)$ if possible

Try I: $y = 2x+\cos(x)$ can't solve for x!

Test for inverse functions: horizontal line test



no inverse



if every horiz. line intersects graph at most once $\Rightarrow$ f has inverse.

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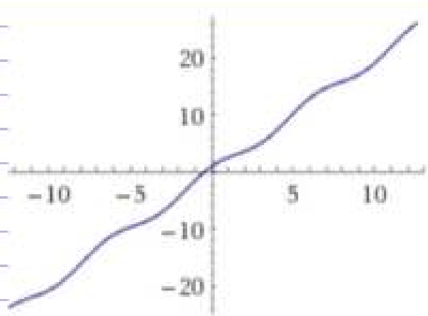
Panel 16

$f(x) = 2x+\cos(x)$



passes vert. line test $\Rightarrow$ has inverse.

what is $f^{-1}(1) = t$ | $f()$

$f(f^{-1}(1)) = f(t)$

$1 = f(t) = f(0)$

$f^{-1}(1) = 0$

want: $(f^{-1})'(1) =$ HW

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