Panel 1

Optimization

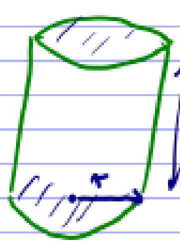
Blah blah... long story... blah maximize (or minimize) blah blah...

- draw picture
- name variables
- add vars. to picture
- set up equations about what you know
- one equation with one variable to max. or min
- find f', critical points, etc. Helpful to have $[0,5]$

1

Panel 2

Ex: A cylindrical can needs to hold 1 l of oil. Find the dimensions that minimize the cost of the metal.



h = height, r = radius

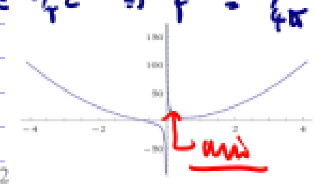
Know: $1 = \text{volume} = \pi r^2 h \Rightarrow h = \frac{1}{\pi r^2}$

Min. surface area: $A = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$, $r \in (0, ?)$

$A'(r) = 4\pi r - \frac{2}{r^2} = 0 \Rightarrow 4\pi r = \frac{2}{r^2} \Rightarrow r^3 = \frac{2}{4\pi} = \frac{1}{2\pi}$

$r = \sqrt[3]{\frac{1}{2\pi}}$

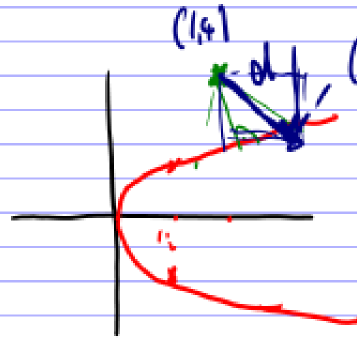


2

Panel 3

Find the point on $y^2 = 2x$ that is closest to $(1,4)$.

minimize distance



$(x,y) = (\frac{1}{2}y^2, y)$

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$
$$= \sqrt{(\frac{1}{2}y-1)^2 + (y-4)^2}$$

Minimizing d and d^2 occurs at same point!

$\Rightarrow$ Minimize $d^2 = (\frac{1}{2}y^2-1)^2 + (y-4)^2 = f(y)$

There must be a min., so $\downarrow$ is it

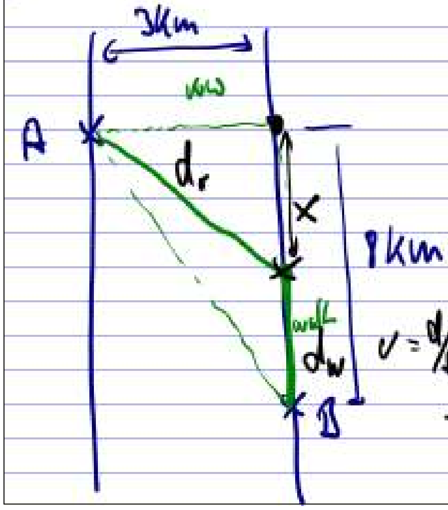
$$f'(y) = 2(\frac{1}{2}y^2-1)\cdot y + 2(y-4)\cdot 1 = y^3 - 2y + 2y - 8$$
$$= y^3 - 8 = 0 \Rightarrow y = 2$$

Min. at $y=2$ is $d = \sqrt{4} = 2$

3

Panel 4

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)



Want: minimize travel time T

$$d_w = 8-x \Rightarrow t_w = \frac{8-x}{8}$$
$$d_r = \sqrt{9+x^2} \Rightarrow t_r = \frac{\sqrt{9+x^2}}{6}$$

$v = d/t \Rightarrow t = d/v$ $\qquad x \in [0,8]$

$$T = t_w + t_R = \frac{1}{6}\sqrt{9+x^2} + \frac{1}{8}(8-x).$$

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Panel 5

Minimize $T = \frac{1}{6}\sqrt{9+x^2} + \frac{1}{8}(8-x)$, $x \in [0,8]$

$$T' = \frac{1}{6}\,\frac{1}{\cancel{2}}(9+x^2)^{-1/2}\cdot \cancel{2}x - \frac{1}{8} = 0$$

$$\frac{1}{6}\,\frac{x}{\sqrt{9+x^2}} - \frac{1}{8} = 0$$

$$\frac{x}{\sqrt{9+x^2}} = \frac{6}{8} = \frac{3}{4}$$

$$x = \frac{3}{4}\sqrt{9+x^2}$$

$$x^2 = \frac{9}{16}(9+x^2) \Rightarrow x^2 - \frac{9}{16}x^2 = \frac{81}{16} \Rightarrow x^2\,\frac{7}{16} = \frac{81}{16}$$

$x^2 = \frac{81}{7}$, $x = \sqrt{\frac{81}{7}} = 3.4$

x	T
0	
8	
3.4	⇐ min

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Panel 6

Business Math

If $C(x)$ is the cost of producing x units

$C'(x)$ is called marginal cost, i.e. cost to produce one more item!

fixed cost = cost to produce nothing

If $p(x)$ is price per unit, then — demand function

$R(x) = x\,p(x)$ Revenue

"marginal" = Business speak for derivative

$P(x) = R(x) - C(x)$

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Panel 7

Ex 1: If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, find the cost and the marginal and fixed cost at a production level of $x = 1000$

$\Rightarrow C(1000) =$ fixed cost: $C(0) = 16000$

$C'(1000) =$ marg.

Ex 2: If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost, and $p(x) = 1{,}700 - 7x$ is the demand function, find production level to max. profit.

$R(x) = x\,p(x) = x \cdot (1700 - 7x)$

$P(x) = R - C = x \cdot (1700 - 7x) - (16000 + 500x - 1.6x^2 + 0.004x^3)$

$P'(x) = 0 \Rightarrow x = \_\_\_$ as usual

Panel 8

Need a break: back to differentiation techniques:

Suppose $x^2 + y^2 = 9$. Find $y'(1)$

Idea 1: solve for y. $y = \pm\sqrt{9 - x^2}$

$$y' = \pm \frac{1}{2}(9 - x^2)^{-1/2} \cdot (-2x) = \mp \frac{x}{\sqrt{9 - x^2}}$$

$$\Rightarrow y'(1) = \mp 1/\sqrt{8}$$

Idea 2: Think of $y = y(x)$ as an unknown function of x:

$$\frac{d}{dx}\left[x^2 + y^2 = 9\right] \Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(9)$$

$$2x + 2y \cdot y' = 0 \Rightarrow y' = -\frac{x}{y} = \mp 1/\sqrt{8}$$

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Panel 9

Implicit Differentiation

$\boxed{y =} g(x)$ explicitly defines y as a function of x.

y' as usual

$f(x,y) = c$ implicitly defines y as a function of x or x as a function of y.

$\Rightarrow$ implicit differentiation

$y' = \frac{3x^2 - 6y}{6x - 3y^2}$

Ex: $x^3 + y^3 = 6xy$ Say $y = y(x)$ - unknown.

$\Rightarrow \frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[6xy]$

$3x^2 + 3y^2 \cdot y' = 6y + 6xy'$

$3x^2 - 6y = 6xy' - 3y^2y'$

$= y'(6x - 3y^2)$

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Panel 10

$x^3 + y^3 = 6xy$, assume $x = x(y)$ - unknown. Find x'

$\frac{d}{dy}(x^3 + y^3) = \frac{d}{dy}(6xy)$

$3x^2 \cdot x' + 3y^2 = 6x'y + 6x \cdot 1$

Ex: Find y' if $\sin(x+y) = y^2\cos(x)$ (i.e. $y = y(x)$)

$\frac{d}{dx}[\sin(x+y)] = \frac{d}{dx}[y^2\cos(x)]$

$\cos(x+y)\cdot(1+y') = 2y\cdot y'\cdot\cos(x) + y^2(-\sin(x))$

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