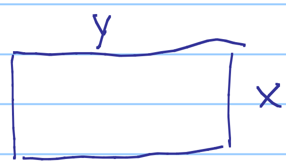
Homework Set # 19. Chapter 4.5 and 2.6

Note Title 10/18/2012

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.



x = height
y = width

$\Rightarrow$ Area $A = xy$

given

Perimeter $P = \boxed{2x+2y = 100} \Rightarrow x+y=50 \Rightarrow y = 50-x$

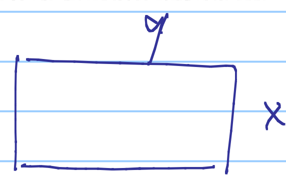
Subst $y = 50-x$ into area: $A = x(50-x) = 50x - x^2$. Clearly, $0 \le x \le 50$.

To max. we find critical points: $A' = 50-2x = 0 \Rightarrow x = 25$. $(\Rightarrow y = 50-25 = 25)$

$A(0) = 0$, $A(25) = 25^2 = 625$, $A(50) = 0$.

$\Rightarrow$ The rect with largest area is a square of side length $\underline{25\,m}$. Max area is $\underline{625\,m^2}$

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.



(as before)

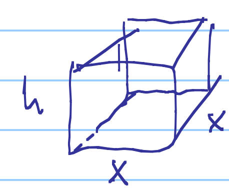
$P = 2x + 2y$ to be minimized

$A = xy = 1000 \Rightarrow y = \frac{1000}{x}$

$\Rightarrow P = 2x + \frac{2000}{x}$ so that $P'(x) = 2 - \frac{2000}{x^2} = 0 \Rightarrow x^2 = 1000 \Rightarrow x = \sqrt{1000} = 31.62$

Thus: $x = 31.62$ and $y = 31.62$ give the smallest perimeter of $P = 126.5$ m.

9. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.



x = side of base square

y = height

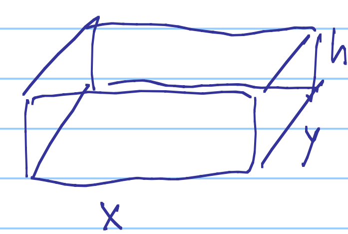
Know: surface area $A = \underbrace{x^2}_{\text{bottom}} + \underbrace{4xh}_{\text{sides}} = 1200$ Want: max $V = x^2 h$

$$\Rightarrow h = \frac{1200 - x^2}{4x} \Rightarrow V = x^2\left(\frac{1200 - x^2}{4x}\right) = \frac{1200x}{4} - \frac{x^3}{4} = 300x - \frac{1}{4}x^3$$

Now as usual: $V'(x) = 300 - \frac{3}{4}x^2 = 0 \Rightarrow \frac{3}{4}x^2 = 300 \Rightarrow x = \pm\sqrt{400} = \underline{\underline{20}}$,

Thus, let $\underline{x = 20}$ and $\underline{h} = \frac{1200 - 20^2}{4 \cdot 20} = \underline{10}$ gives box with largest volume $\underline{V = 4000 cm^2}$

12. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.



x = length
y = width
h = height

Know: $x = 2y$ $\qquad V = xyh = \boxed{(2y)y \cdot h = 10} \Rightarrow h = \frac{10}{2y^2} = 5/y^2$

Cost of base: $\$10 \cdot x \cdot y = 10(2y)y = 20y^2$

Cost of sides: $\$6 \cdot 4xh = 24(2y) \cdot 5/y^2 = 120/y$

$\Rightarrow$ Total cost: $C(y) = 20y^2 + 120/y$

$\Rightarrow C'(y) = 40y - \frac{120}{y^2} = 0 \iff 40y^3 - 120 = 0 \Rightarrow y^3 = 3 \Rightarrow y = \sqrt[3]{3} = 1.73$ m

Thus: $\underline{\underline{y = 1.73}}$, $\underline{\underline{x}} = 2 \cdot y = \underline{\underline{3.46}}$, $\underline{\underline{h}} = 5/y^2 = \underline{\underline{1.67\text{ m}}}$ give cheapest desired container.

34. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

Slope of tangent is

$y' = 120x^2 - 15x^4$

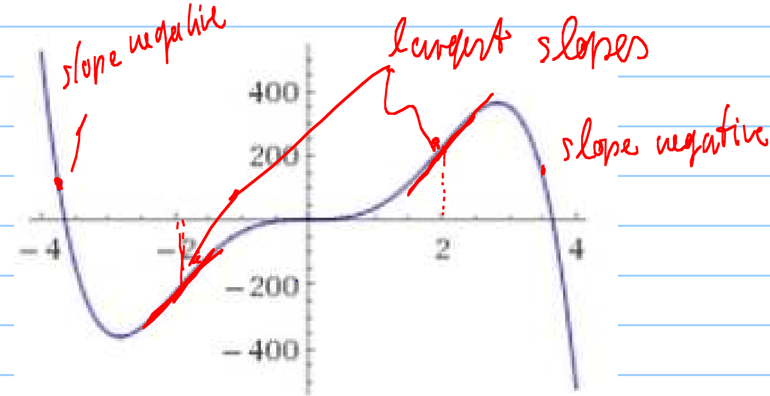
So want to max. $q(x) = 120x^2 - 15x^4$

$\Rightarrow q'(x) = 240x - 60x^3 = 60x(4-x^2) = 0 \quad \Rightarrow x = 0, \pm 2$ are all critical!

$q(0) = 0$, $q(2) = 120 \cdot 2^2 - 15 \cdot 2^4 = 120 \cdot 4 - 15 \cdot 16 = 480 - 240 = \underline{\underline{240}}$

$q(-2) = 240$

$\Rightarrow x = \pm 2$ give the largest tangent slope of 240.



36. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.

(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

$P(x) = R(x) - C(x)$

revenue − cost

a) P has a max $\Rightarrow P'(x) = 0 = R'(x) - C'(x)$

$\Rightarrow C'(x) = R'(x)$, i.e. marginal revenue = marg. cost

b) $p(x) = 1700 - 7x \Rightarrow R(x) = xp(x) = x(1700 - 7x)$

$\Rightarrow P(x) = R(x) - C(x) = 1700x - 7x^2 - (16000 + 500x - 1.6x^2 + 0.004x^3) =$

$= -16000 + 1200x - 5.4x^2 - 0.004x^3$

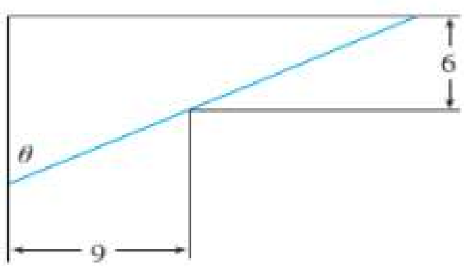
$\Rightarrow P'(x) = 1200 - 10.8x - 0.012x^2 = 0$

$-0.012(x - 100)(x + 1000) = 0 \quad \Rightarrow x = 100$ and $x = -1000$

can't have neg. units

Thus, max profit occurs at production level of $\underline{x = 100}$ units

46. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



I am not sure. We'll tackle this one later

Section 2.6

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3. $x^3 + x^2y + 4y^2 = 6$

Find dy/dx, i.e. y is a function of x and the chain rule applies to y (but not to x).

$$\frac{d}{dx}\left(x^3 + x^2 y + 4y^2\right) = \frac{d}{dx}(6)$$

$$3x^2 + 2x \cdot y + x^2 \cdot y' + 8y \cdot y' = 0$$ ← this is a good answer

$$y'(x^2 + 8y) = -3x^2 - 2xy$$

$$\rightarrow y' = \frac{-3x^2 - 2xy}{x^2 + 8y}$$ ← this works as well

7. $x^2y^2 + x\sin y = 4$

$$\frac{d}{dx}\left(x^2y^2 + x\sin(y)\right) = \frac{d}{dx}(4)$$

$$2xy^2 + x^2 \cdot 2yy' + 1 \cdot \sin(y) + x\cos(y)\,y' = 0$$ ← this is fine

$$y'\left(2x^2y + x\cos(y)\right) = -2xy^2 - \sin(y)$$

$$y' = \frac{-2xy^2 - \sin(y)}{2x^2y + x\cos(y)}$$

9. $4 \cos x \sin y = 1$

$$\frac{d}{dx}\left[4\cos(x)\sin(y)\right] = \frac{d}{dx}(1)$$

$$-4\sin(x)\cdot\sin(y) + 4\cos(x)\cos(y)\,y' = 0$$ ← this is fine

$$y' = \frac{4\sin(x)\sin(y)}{4\cos(x)\cos(y)} = \tan(x)\tan(y)\ !$$

17–22 ▪ Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

17. $x^2 + xy + y^2 = 3$, $(1, 1)$ (ellipse)

tangent line $y = mx + b$ where m is slope of tangent ⇒ need y'. So

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3) \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow \text{at } (1,1), \text{ i.e. } x = , y = 1,$$

we have: $2\cdot 1 + 1 + 1y' + 2\cdot 1\cdot y' = 0$

$3 + 3y' = 0 \Rightarrow y' = -1$ is the slope at $(1,1)$