Panel 1

Math 1501: Last time

Formal def. of: $\lim_{x \to a} f(x) = L$

Given any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$

Squeezing theorem $g(x) \le f(x) \le h(x)$

$g(x) \to L$, $f(x) \to L$, $h(x) \to L$ as $x \to a$

eg. $x \sin\left(\frac{\pi}{x}\right)$

Special limits:

$$\lim_{x \to 0} \frac{\sin(2x)}{\cos(5x)} = 0$$

$$\lim_{x \to 0} \frac{\sin(2x)}{\sin(5x)} = \lim \frac{\frac{\sin(2x)}{2x}\,2x}{\frac{\sin(5x)}{5x}\,5x} = \frac{2}{5}$$

$\frac{\sin(2x)}{2x} \to 1$, $\frac{\sin(5x)}{5x} \to 1$

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Panel 2

Tricky:

$$\lim_{x \to 0} \frac{|x|}{x} = \text{d.n.e.}$$

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} 1 = 1$$

$$\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{if } x > 0 \\ \frac{-x}{x} = -1 & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^-} -1 = -1$$

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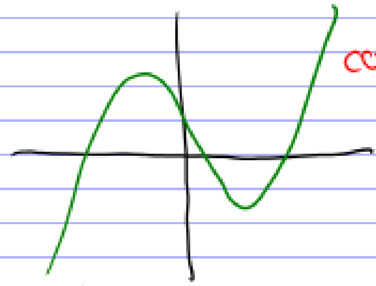
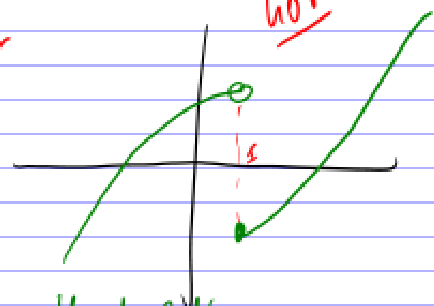
Panel 3

Application of Limits: Continuity



cont. ✓



not cont. at $x=1$

If you can draw graph without lifting pen => cont.!

Def: A function $f(x)$ is continuous at $x=a$ if:

$\lim\limits_{x\to a} f(x) = f(a)$

(1) $f(a)$ exists

(2) $\lim\limits_{x\to a} f(x)$ exist

(3) both are equal

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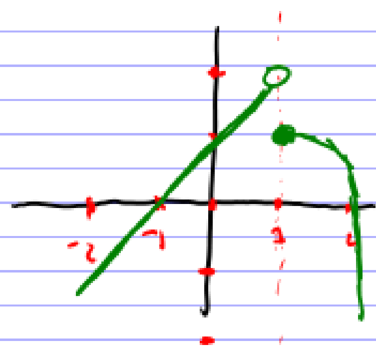
Panel 4

Example:



$\lim\limits_{x\to 1} f(x) = \text{d.n.e}$

$f(1)=1$

not cont. at $x=1$

Def: $\lim\limits_{x\to a} f(x) = f(a)$ ✓

$g(x) = \dfrac{x^2-4}{x-2}$

Is g cont. at $x=2$? $g(2) = \underline{\text{d.n.e}}$

No!

$h(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ \cancel{7}\ 4 & \text{if } x = 2 \end{cases}$

$h(2) = 7$

$\neq$

Not cont.

$\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{x-2} = 4$

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Panel 5

Continuity is easy if you see graph of $f(x)$.
Otherwise: continuity means to check 3 conditions

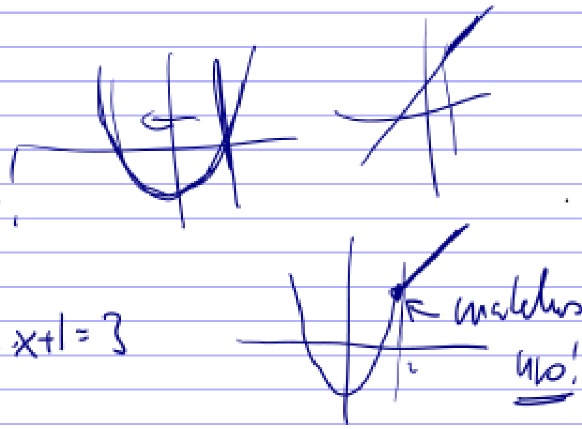
Ex: $f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 2 \\ x + 1 & \text{if } x > 2 \end{cases}$ cont. at $x = 2$?!?

~~$f(2)$ d.n.e.~~

$f(2) = 3$

$\lim_{x \to 2} f(x) = 3$ yes!

$\lim_{x \to 2^-} x^2 - 1 = 3$, $\lim_{x \to 2^+} x + 1 = 3$



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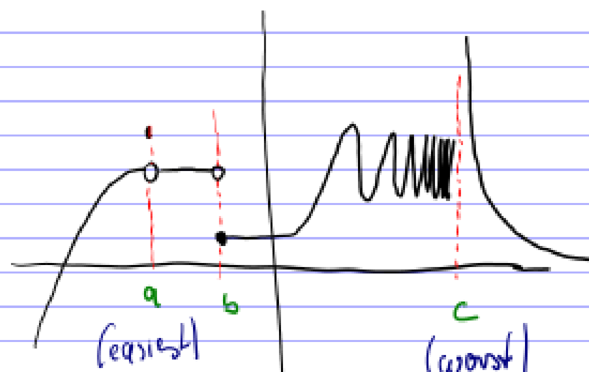
Panel 6

Dis-Continuity Types

3 ways in which a function can fail to be cont.:



removable: if $\lim_{x \to a} f(x)$ exists

jump: if $\lim_{x \to a^+} f(x)$ exist and $\lim_{x \to a^-} f(x)$ exist but are different

essential: otherwise

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Panel 7

Ex: Identify and classify the discontinuities

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ \cancel{3}\ 1 & \text{if } x = 0 \end{cases}$$

removable!

$$g(x) = \begin{cases} x^2 - 1 & \text{if } x < 1 \\ x + 2 & \text{if } x > 1 \end{cases}$$

$\lim_{x \to 1^-} g = 0$

$\lim_{x \to 1^+} g = 3$

jump

$$h(x) = \begin{cases} x^2 + 2 & \text{if } x < 0 \\ x + 2 & \text{if } x > 0 \end{cases}$$

$\lim_{x \to 0^-} = 2$

$\lim_{x \to 0^+} = 2$

removable d.n.e.

$$k(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 15 & \text{if } x = 0 \end{cases}$$

essent.

$\lim_{x \to 0^-} k(x) = \lim_{x \to 0^+} \sin\left(\frac{1}{x}\right) =$

$\lim_{x \to 0^+} k(x) =$ d.n.e.

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Panel 8

$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ $\qquad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

Nothing helps!

x	$\sin\left(\frac{1}{x}\right)$
$\frac{1}{\pi}$	0
$\frac{1}{2\pi}$	0
$\frac{1}{3\pi}$	0
$\frac{1}{4\pi}$	0
$\frac{1}{\vdots}$	$\vdots$

BUT

x	$\sin\left(\frac{1}{x}\right)$
$\frac{2}{\pi}$	1
$\frac{2}{3\pi}$	-1
$\frac{2}{5\pi}$	1
$\frac{2}{7\pi}$	-1
$\frac{2}{\vdots}$	+1
$\frac{\vdots}{0}$	-1

no pattern as $x \to 0$

$\Rightarrow$ no limit

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Panel 9

Theorems about Continuity:

f, g cont. $\Rightarrow$ $f+g$, $f-g$, $f \cdot g$ is cont.

f/g is cont. as long as $g(x) \neq 0$

$(f \circ g) = (f(g(x)))$ is cont.

Thm: polynomials are cont. for all x

rational functions are cont. where defined

roots are cont. where defined

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Panel 10

On which intervals are the following functions continuous:

$f(x) = x^{100} - 2x^{77} + 75$ cont.

$g(x) = \frac{x^2 + 2x + 17}{x^2 - 1}$ cont. except for $x = \pm 1$

$h(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2-4}$ cont. except for $x = 1, 2, -2$

cont.: $[0,1) \cup (1,2) \cup (2,\infty)$ and x must be ≥ 0

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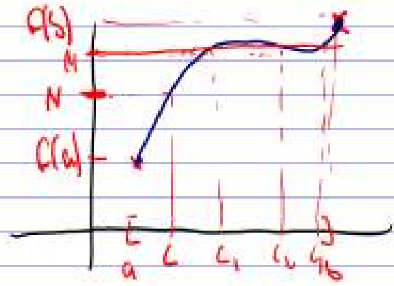
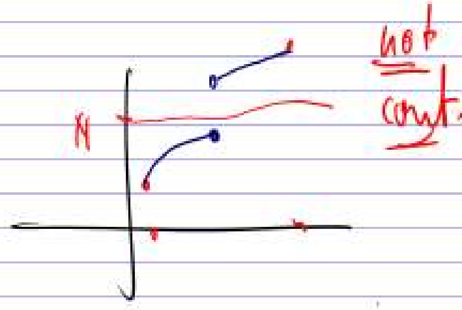
Panel 11

Intermediate Value Theorem

Say f is a cont. function on $[a,b]$. Take any number N with $f(a) < N < f(b)$. Then there is an c such that $f(c) = N$





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Panel 12

Application: Show that

$$f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$$

has a solution between 1 and 2

$f(x) = 4x^3 - 6x^2 + 3x - 2$

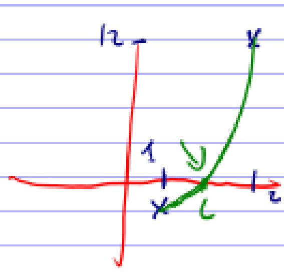
$f(1) = -1$

$f(2) = 32 - 24 + 6 - 2 = 12$

$f(1) = -2$, $f(2) = 12$

$\Rightarrow$ take 0 between -2 and 12 $\Rightarrow$

there is c with $f(c) = 0$



To find answer, half interval $\Rightarrow x = 1.5$

$f(1.5) = 2.5 \Rightarrow$ solution between 1 and 1.5

Again $f(1.25) = 0.1875 \Rightarrow$ sol. between 1 and 1.25

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