Panel 1

Last time:

Maple ✓

Indef integrals

Fund. Theorem of Calc. (1)

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

(if f cont. on [a, b])

Substitution Method

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

Memorize

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Panel 2

Subst. Method

1. Identify subst. $u = \ldots$
2. Take deriv. $\frac{du}{dx} = \ldots$;
3. Solve for $du = \ldots\,dx$ (or similar)
4. Perform subst. and eliminate original variable completely
5. Integrate new problem
6. Re-substitute for final answer.

ex: $\int x^3 \cos(\underbrace{x^4+1}_{u})\,dx$

1.) $u = x^4 + 1$

2.) $\frac{du}{dx} = 4x^3$

3.) $du = 4x^3\,dx$

$\frac{1}{4}\,du = x^3\,dx$

4.) $\int \frac{1}{4}\cos(u)\,du$

5.) $\frac{1}{4}\sin(u) + C$

6.) $\frac{1}{4}\sin(x^4+1) + C$

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Panel 3

$$\int \frac{\sin(x)}{1-\sin^2(x)}\,dx = \int \frac{\sin(x)}{\cos^2(x)}\,dx = \int \frac{\sin}{\cos}\cdot\frac{1}{\cos}\,dx = \int \sec\cdot\tan$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$-\int \frac{1}{u^2}\,du = u^{-1} + C = \frac{1}{\cos(x)} + C = \sec(x) + C$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$\int \cancel{u}$?

$u = \cos^2(x)$

$du = 2\cos(x)\left(-\sin(x)\right)dx$

$u = \cancel{\frac{\sin(x)}{\cos^2(x)}}$

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Panel 4

More Examples

$$\int x^2(x^3+2)\,dx = \int x^5 + 2x^2\,dx = \frac{1}{6}x^6 + \frac{2}{3}x^3 + C$$

$$\int x^2(x^3+2)^2\,dx$$

- foil + work it out...
- $u = x^3 + 2$ .....

$$\int x^2 (x^3+2)^{10}\,dx = \int \frac{1}{3} u^{10}\,du = \frac{1}{3}\cdot\frac{1}{11}u^{11} + C = \frac{1}{33}(x^3+2)^{11} + C$$

$u = x^3 + 2$

$du = 3x^2\,dx$

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Panel 5

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\,\frac{2}{3}\,u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C.$$

$u = 2x+1$

$du = 2\,dx$

Works for any linear substitution

$$\int \frac{1}{(7x+9)^3}\,dx \sim \int \frac{1}{u^3}\,du$$

$$\int x\sqrt{x-1}\,dx = \int x\sqrt{u}\,dx = \int (u+1)\sqrt{u}\,du =$$

$u = x-1 \Rightarrow u+1 = x$!

$du = dx$

$$= \int u^{3/2} + u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

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Panel 6

Substitution for Definite Integrals

1.) $u = \ldots$

2.) $du = \ldots$

3.) Rewrite integral
BUT KEEP ORIGINAL
BOUNDS

4.) Integrate + Resubstitute

5.) Plug in original bounds

OR: in (3) change bounds
in (4) no resubst. necessary

Ex: $\int_0^2 (x+1)^{20}\,dx$

$u = x+1$, $du = dx$

$$\int_{x=0}^{x=2} u^{20}\,du = \frac{1}{21}u^{21}\Big|_{x=0}^{x=2} =$$

$$= \frac{1}{21}(x+1)^{21}\Big|_0^2 =$$

$$= \frac{1}{21}(3)^{21} - \frac{1}{21}$$

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Panel 7

$$\int_0^{2} (x+1)^{20}\,dx = \int_1^3 u^{21}\,du = \frac{1}{21} u^{21}\Big|_1^3 = \frac{1}{21}(3)^{21} - \frac{1}{21}$$

$u = x+1$

$du = dx$

if $x=0 \Rightarrow u=1$

if $x=2 \Rightarrow u=3$

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx = \begin{cases} -\frac{1}{5}\int_{x=1}^{x=2} \frac{1}{u^2}\,du = \text{Resubstitute} \\ -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du = \text{work out without resubst.} \end{cases}$$

$u = 3-5x$

$du = -5\,dx$

if $x=1 \Rightarrow u=-2$, $x=2 \Rightarrow u=-7$

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Panel 8

$$\int_{-1}^{1} 2x \sin^5(x^2)\,dx = \begin{cases} a) \int_{x=-1}^{x=1} \sin^5(u)\,du \quad \left(= \cancel{\int_{-1}^{1} \sin^5(u)\,du} \right) \\ b) = \int_1^1 \sin^5(u)\,du = 0 \end{cases}$$

$u = x^2$

$du = 2x\,dx$

if $x=-1 \Rightarrow u=1$

if $x=1 \Rightarrow u=1$

$f(x) = \frac{1}{6}\sin^6(x)$

$\Rightarrow f'(x) = \frac{1}{\cancel{6}} \cdot \cancel{6} \sin^5(x) \cdot \boxed{\cos(x)}$

Subst. bounds can be easier, so: don't forget!

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Panel 9

$$\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx = \frac{1}{4}\int_1^9 \frac{1}{\sqrt{u}}\,du = \frac{1}{4}\cdot 2u^{1/2}\Big|_1^9 = \frac{1}{2}(9^{1/2}) - \frac{1}{2}\,1$$

$u = 2x^2+1$;
$du = 4x\,dx$,

if $x=0 \Rightarrow u=1$
if $x=2 \Rightarrow u=9$

Hard $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2x^2+1}}\,dx$

Strange $\displaystyle\int_0^8 \frac{x}{\sqrt{1+x}}\,dx = \int_{x=0}^{x=8} \frac{u-1}{\sqrt{u}}\,du = \int_{x=0}^{x=8} u^{1/2} - u^{-1/2}\,du = HW$

$u = 1+x$

$du = dx$

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Panel 10

Bounds can be variables:

$$F(x) = \int_0^x t^2\,dt = \frac{1}{3}t^3\Big|_0^x = \frac{1}{3}x^3 - 0$$

$\Rightarrow F'(x) = x^2$

Fundamental Theorem of Calculus, part 2

If f is continuous on $[a,b]$ then

$F(x) = \int_a^x f(t)\,dt$ is a function of x s.t.

$F'(x) = f(x)$, i.e. $\cancel{\frac{d}{dx}}\int_a^x f(t)\,dt$

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Panel 11

What is: $\cancel{\frac{d}{dx}}\cancel{\int_0^{x}}\sqrt{1+t^2}\,dt = \sqrt{1+x^2}$

$$\frac{d}{dx}\int_0^{15}\sqrt{1+t^2}\,dt = 0$$

Why? $B(x) = \int_0^{x}\sin(t^2)\,dt = ?$ Has no anti derivative!

Graph it anyway: $B'(x) = \cancel{\frac{d}{dx}}\cancel{\int_0^{x}}\sin(t^2)\,dt = \sin(x^2)$

$x = 0$ is critical. $\Rightarrow B(0) = \int_0^0 \sin(t^2)\,dt = 0$

$B''(x) = 2x\cos(x^2)$ could graph it

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Panel 12

Properties of the Integral

$\int_a^a f(x)\,dx = 0$, $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

If $f \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$ (area)

If f is pos. + neg. $\Rightarrow \int_a^b f(x)\,dx$ is "signed" net area

$$\int c\cdot f(x) + d\,g(x)\,dx = c\int f(x)\,dx + d\int g(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

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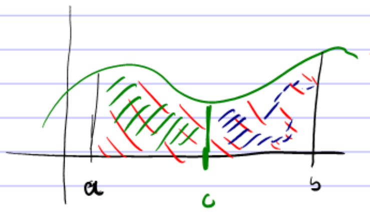
Panel 13



$$\int_a^b f(x)\,dx = \text{red area}$$

$$\int_a^c f(x)dx = \text{green}, \quad \int_c^b f(x)dx = \text{blue}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

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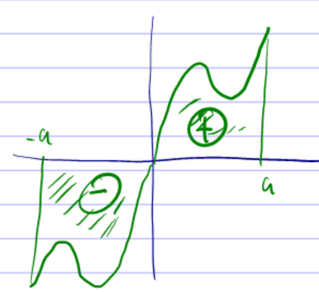
Panel 14

$f(-x) = -f(x)$

If $f(x)$ is <u>odd</u> then $\int_{-a}^{a} f(x)\,dx = 0$

<u>Proof:</u> $\int_{-a}^{a} f(x)dx =$

$$\int_{-a}^{0} f(x)dx + \boxed{\int_0^a f(x)dx} = 0$$



let $u = -x$, if $x = -a \Rightarrow u = a$

if $x = 0 \Rightarrow u = 0$

$$-\int_a^0 f(-u)\,du = \int_0^a f(-u)du = -\boxed{\int_0^a f(u)du}$$

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Panel 15

HW: If f is even then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Prove that (extra credit)

Section 4.5

Happy Thanksgiving!

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