Panel 1

Last time: Anti-Derivatives

If $f'(x) = x^3 - \sqrt{x} + \sec(x)\tan(x)$ ($\sqrt{x} = x^{1/2}$)

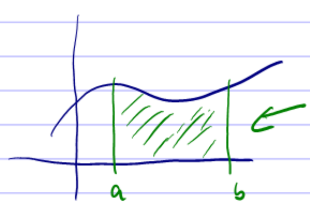
then $f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \sec(x) + C$

Right because $f'(x)$ is what it should be!

1

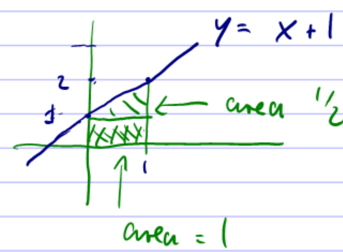
Panel 2

New topic: find area under a curve!



← what is this area?

Sometimes I know:



area is $1\frac{1}{2} = \underline{\underline{3/2}}$ only for simple functions

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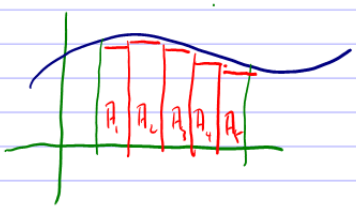
Panel 3

Want:

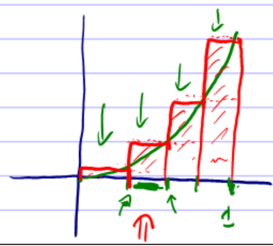


Idea: approximate area by using rectangles

$A \approx A_1 + A_2 + A_3 + A_4 + A_5$

Use more rectangles for better approx.

Ex: Approx. area under $f(x) = x^2$ on $[0,1]$ using 4 rectangles



$A = A_1 + A_2 + A_3 + A_4 =$

$= \frac{1}{4} \cdot f(\frac{1}{4}) + \frac{1}{4} f(\frac{2}{4}) + \frac{1}{4} f(\frac{3}{4}) + \frac{1}{4} f(1)$

$= \frac{1}{4} \left((\frac{1}{4})^2 + (\frac{2}{4})^2 + (\frac{3}{4})^2 + (\frac{4}{4})^2 \right) =$

pick right side of each sub interval

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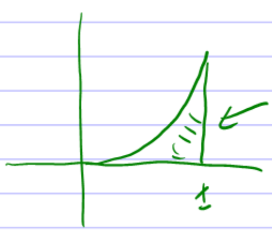
Panel 4

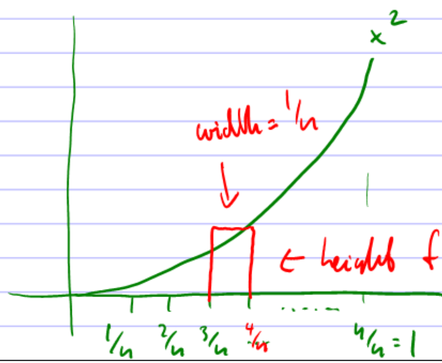
$\frac{1}{4} \left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1 \right) = \frac{1}{4} \cdot 1.875 = \underline{\underline{0.46}}$



Area is about 0.46

(actually it's less than that)

To find better approx, use n rectangles.



$A_4 = \frac{1}{n} \cdot f(\frac{4}{n})$

k-th rectangle

$A_k = \frac{1}{n} f(\frac{k}{n})$

4

Panel 5

$$A \approx \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \dots + \frac{1}{n} f\left(\frac{n}{n}\right), \quad f(x) = x^2$$

$$= \frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right) =$$

$$= \underbrace{\frac{1}{n} \cdot \frac{1}{n^2}}\underbrace{\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)} =$$

Fact: $= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$

Thus: Area $\approx \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$

Now take bigger n and bigger n to get area exactly

$$A = \lim_{n \to \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = 2/6 = 1/3 \approx \underline{0.3333}$$

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Panel 6

Fact:

$$\begin{array}{l} 1 + 2 + 3 + 4 + \dots + (n-1) + n = \\ n + (n-1) + (n-2) + \dots + 2 + 1 \\ \hline (n+1) + (n+1) + \dots (n+1) + (n+1) = n(n+1) \end{array}$$

(A) $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$ ✓

(B) $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ faith

(C) $1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ faith

Ex: $1 + 2 + 3 + \dots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = \underline{5050}$

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Panel 7

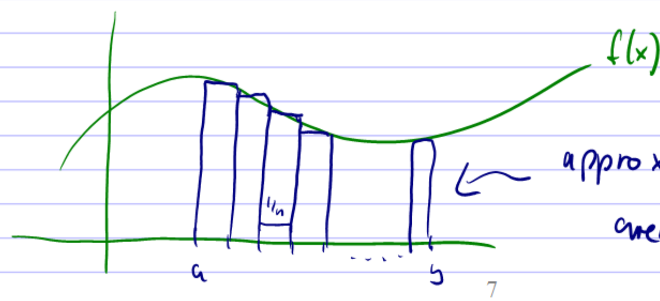
Definition: Take a positive function f on $[a,b]$.

Divide $[a,b]$ into n equal pieces and define

$$R_n = \frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \dots + \frac{1}{n} f(x_n)$$

Then area under f from a to b is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \dots + \frac{1}{n} f(x_n)$$



approximates the area under $f(x)$

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Panel 8

Consider $f(x) = 2x$, where $x \in [0,1]$

a) Approx area under f using 5 subdivision

b) Find exact area under $f(x)$

$$A \approx \frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \dots + \frac{1}{n} f(x_n)$$

a) $n = 5$: $A \approx \frac{1}{5} f\left(\frac{1}{5}\right) + \frac{1}{5} f\left(\frac{2}{5}\right) + \frac{1}{5} f\left(\frac{3}{5}\right) + \frac{1}{5} f\left(\frac{4}{5}\right) + \frac{1}{5} f\left(\frac{5}{5}\right) =$

$$= \frac{1}{5}\left(2\frac{1}{5} + 2\frac{2}{5} + 2\frac{3}{5} + 2\frac{4}{5} + 2\cdot\frac{5}{5}\right) =$$

$$= \frac{1}{5}\cdot\frac{2}{5}\left(1+2+3+4+5\right) = \frac{30}{25}$$

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Panel 9

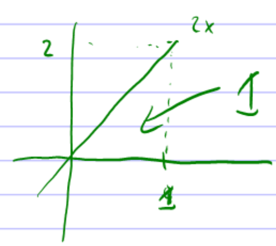
5) n : $A \approx \frac{1}{n} f(\frac{1}{n}) + \frac{1}{n} f(\frac{2}{n}) + \frac{1}{n} f(\frac{3}{n}) + \frac{1}{n} f(\frac{4}{n}) + \frac{1}{n} f(\frac{5}{n}) \dots \frac{1}{n} f(\frac{n}{n})$

$= \frac{1}{n} \left(2 \frac{1}{n} + 2 \frac{2}{n} + \dots + 2 \cdot \frac{n}{n} \right) =$

$= \frac{1}{n} \cdot \frac{2}{n} \left(1 + 2 + 3 + \dots + n \right) = \frac{1}{n} \cdot \frac{\cancel{2}}{n} \cdot \frac{n(n+1)}{\cancel{2}}$

$A = \lim_{n \to \infty} \frac{n(n+1)}{n^2} = 1$

Found Area under $f(x) = 2x$ on $[0, 1]$ is 1.



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Panel 10

Find area under $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$

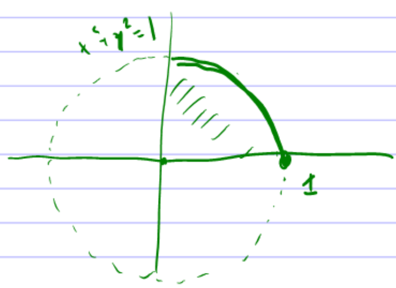
$\Uparrow$

$y = \sqrt{1 - x^2} \quad | ( )^2$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

Area is $\pi/4$



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Panel 11

Definition: We defined the integral of a function f on $[a,b]$ as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \dots + \frac{1}{n} f(x_n)\right)$$

integral of f from a to b

Note: $\int_a^b f(x)dx$ is area under $f(x)$ from a to b if f is positive

Ex: $\int_0^1 2x\,dx = 1$, $\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$

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Panel 12

Derivative	Integral
funky def:	funky def:
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$	$\int_a^b f(x)dx = \lim_{n\to\infty} R_n(x)$
geometrically:	geometrically:
slope of tangent	area under curve / function (if f positive)
how to do it.	how to do it
prod., quot., chain.	Evaluation theorem

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Panel 13

Evaluation Theorem: f cont. on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of f

Ex: $\int_0^1 2x\,dx = F(1) - F(0) = 1^2 - 0^2 = 1$

If $f(x) = 2x \Rightarrow F(x) = x^2$

Ex: $\int_0^{2\pi} \sin(x)\,dx = -\cos(2\pi) + \cos(0) =$

$= -1 + 1 = 0$

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Panel 14

$\int_0^{\pi/4} 10x^4 + 2\sec^2(x)\,dx = F(\pi/4) - F(0) = 2\left(\frac{\pi}{4}\right)^5 + 2 - 0$

$f(x) = 10x^4 + 2\sec^2(x)$

$\Rightarrow F(x) = 2x^5 + 2\tan(x)$

$0 \qquad 0$

$= 2\cdot\left(\frac{\pi}{4}\right)^5 + 2$

HW as posted tonight!

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