Panel 1

The Derivative

Definition

Graphical Interpretation:

Rules:

Example: $f(x) = \left(2\sqrt{x} + \frac{2}{\sqrt{x}}\right)^2$

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Panel 2

The Derivative

Definition $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

Graphical Interpretation: slope of tangent

Rules: Power Rule: $(x^p)' = p\,x^{p-1}$

→ Constant: $(c \cdot f(x))' = c\,f'(x)$, $\frac{d}{dx}(c) = 0$

Example: $f(x) = \boxed{\left(2\sqrt{x} + \frac{2}{\sqrt{x}}\right)^2} = 4x + 8 + \frac{4}{x} = 4x + 8 + 4x^{-1}$

$(x)^{1/2}$ ✓

$\cancel{(3x+1)^{1/2}}$

$f'(x) = 4 - 4x^{-2}$

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Panel 3

Derivatives of sin and cos

Recall: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ — trig ID

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(x)-1}{x} = 0$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0} \cos(x)\frac{\sin(h)}{h} =$$

$$= \sin(x)\cdot 0 + \cos(x) = \cos(x)$$

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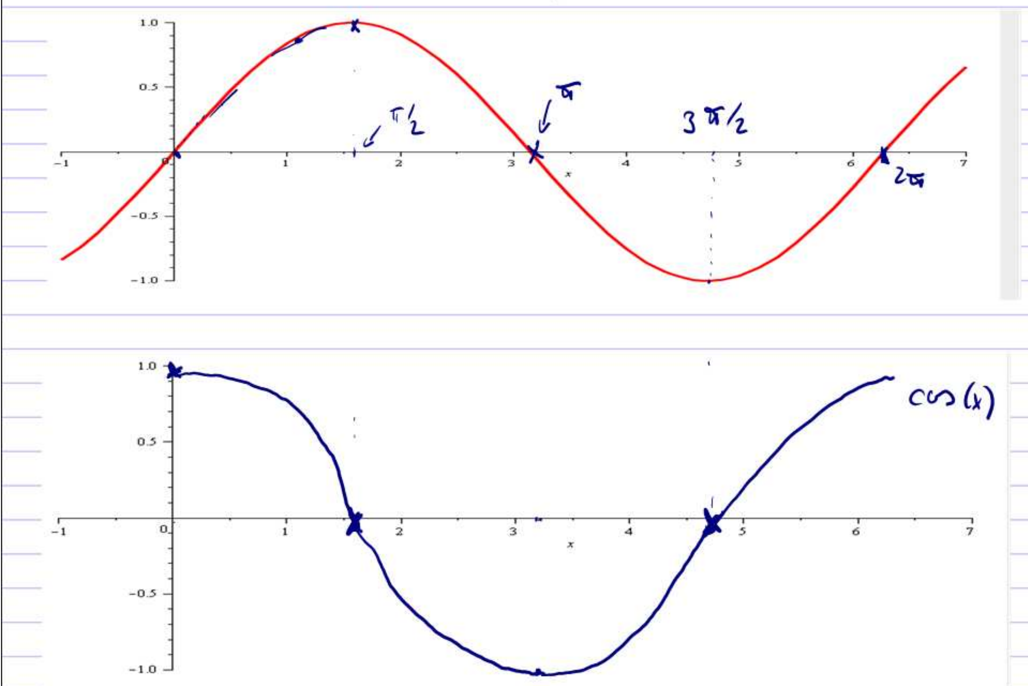
Panel 4

Derivative of sin graphically



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Panel 5

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$ prove this as HW

Ex: $f(x) = \sin(x) - 3\cos(x) + \sin(\pi^2)$

$f'(x) = \cos(x) + 3\sin(x) + 0$

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Panel 6

Product Rule

Ex: $x^4 = (x^2)(x^2) \Rightarrow$ deriv. $(2x)(2x) = 4x^2$

$\Rightarrow$ deriv. $4x^3$

Deriv. of Product ≠ product of derivatives

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$= 0$

$$= \lim_{h\to 0}\frac{g(x+h)\left(f(x+h)-f(x)\right)}{h} + \lim_{h\to 0}\frac{f(x)\left(g(x+h)-g(x)\right)}{h} =$$

by cont.

$$= \lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\frac{g(x+h)-g(x)}{h}$$

$g(x)$ $f'(x)$ $g'(x)$

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Panel 7

Product Rule:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$$

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Panel 8

Product Rule Examples

$$f(x) = \sin(x)\cos(x) \Rightarrow f'(x) = \cos(x)\cdot\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x)$$

$$g(x) = x^2(x^4+3x) = x^6 + 3x^3 \Rightarrow f'(x) = 6x^5 + 9x^2$$

$$\Rightarrow f'(x) = 2x(x^4+3x) + x^2(4x^3+3) = 2x^5 + 6x^2 + 4x^5 + 3x^2 = 6x^5 + 9x^2$$

$$h(x) = \sin^2(x) = \sin(x)\cdot\sin(x), \quad f'(x) = \cos(x)\sin(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x)$$

$$k(x) = \sqrt{x}\cos^2(x) = x^{1/2}\cos(x)\cdot\cos(x)$$

$$h'(x) = \frac{1}{2}x^{-1/2}\cos(x)\cdot\cos(x) + x^{1/2}\left[-\sin(x)\cos(x) + \cos(x)(-\sin(x))\right]$$

$$= \frac{1}{2}x^{-1/2}\cos^2(x) - 2x^{1/2}\sin(x)\cos(x)$$

Panel 9

The Quolient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

$$\lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\cdot g(x)}}{h} =$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{h\cdot g(x+h)\,g(x)} =$$ miracle happens

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

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Panel 10

$$\frac{d}{dx}\frac{hi}{ho} = \frac{ho\, D hi - hi\, D ho}{ho^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$ → Quotient

$$(f\cdot g)' = f'g + fg'$$

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Panel 11

Quotient Rule Examples

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$f(x) = \frac{3x^2 - 2x + 1}{x} \Rightarrow f'(x) = \frac{(6x-2)x - (3x^2-2x+1)\cdot 1}{x^2}$$

or divide out

$$g(x) = \frac{x+1}{x-1} \qquad g'(x) = \frac{1\cdot(x-1) - (x+1)\cdot 1}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

$$h(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$h'(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2(x)$$

$$k(x) = \frac{\sin(x)(x^3+3)}{2-x^2}$$

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Panel 12

$$k(x) = \frac{\sin(x)(x^3+3)}{2-x^2}$$

$$k'(x) = \frac{\left(\cos(x)(x^3+3) + \sin(x)\cdot 3x^2\right)(2-x^2) - \left(\sin(x)(x^3+3)\right)(-2x)}{(2-x^2)^2}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \tan(x)\sec(x)$$

$$= \frac{0\cdot\cos(x) + \sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}$$

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Panel 13

Chain Rule

use definition to prove it!

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

("Onion" rule - one skin at a time)

$f(x) = \sin(3x)$, $f'(x) = \cos(3x) \cdot 3$

$h(x) = (x^2-3)^2$ - (a) square it out)

(b) product rule)

(c) chain rule)

$h'(x) = 2(x^2-3) \cdot 2x$

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Panel 14

$h(x) = \sin(\cos(x^2))$

$\Rightarrow h'(x) = \cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x$

$k(x) = \dfrac{x\cos(3x^2)}{(x^2+1)^3}$

$B = 3(x^2+1)^2 \cdot 2x$

$k'(x) = \dfrac{A\,(x^2+1)^3 - (x\cos(3x^2))[\;B\;]}{(x^2+1)^6}$

$A = \cos(3x^2) + x(-\sin(3x^2)) \cdot 6x$

$(x^p)^q = x^{p \cdot q}$

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Panel 15

Diff. Rules

$(x^p)' = p x^{p-1}$, $(c f(x))' = c f'(x)$, $(c)' = 0$

! Product: $\frac{d}{dx}(f \cdot g) = f'g + fg'$

! Quotient: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

! Chain: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

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Panel 16

#4) $f(x) = x^2 \cos(x)$

$f'(x) = \underbrace{2x} \cdot \underbrace{\cos(x)} + \underbrace{x^2} \cdot \underbrace{(-\sin(x))}$

#7) $f(x) = \frac{\sin(x)}{x^4 - 3}$

$f'(x) = \frac{(\cos(x))(x^4-3) - (\sin(x))(4x^3)}{(x^4-3)^2}$

#8) $f(x) = \sin(\overline{x^3 - 1})$

$f'(x) = \underbrace{\cos(x^3-1)} \cdot \underbrace{3x^2}$

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Panel 17

#11) $f(x) = x^2 \sin(x^3 - 1)$

$$f'(x) = 2x \sin(x^3-1) + x^2 \cdot \left[\cos(x^3-1) \cdot 3x^2\right]$$

#18) $f(x) = \dfrac{x^2 \cos(x)}{(1-2x)^2} = x^2 \cos(x) \left[1-2x\right]^{-2}$

$$f'(x) = \frac{\left(2x\cos(x) + x^2 \cdot (-\sin(x))\right) \cdot (1-2x)^2 - x^2\cos(x) \cdot \left[2(1-2x)^1 \cdot (-2x)\right]}{(1-2x)^4}$$

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Panel 18

Mixed Examples

- from worksheet (see our homepage)

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Panel 19

Applications of Derivative - Physics

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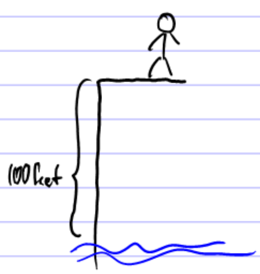
Panel 20

Speed and Acceleration



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Panel 21

Quotient Rule is superfluous:

$$\left(\frac{f}{g}\right) = f \cdot \underbrace{(g)^{-1}}$$

Quotient = Product +

pick up next time

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