Panel 1

Last time: Derivative

Definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Graphically: slope of tangent

New

Notation: $f'(x)$ or $\frac{df}{dx}$ or $\frac{d}{dx} f(x)$

"f prime" df over dx

Alternate Definition:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$

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Panel 2

Quiz

$$\boxed{\lim_{x \to \infty}} \frac{3 - x^2}{1 + x + 2x^2} = -\frac{1}{2}$$

$$\lim_{\boxed{x \to 2^-}} \frac{3x + 1}{x - 2} = \text{"}\frac{+}{-0}\text{"} = -\infty$$

$\frac{0}{-0}$ more work

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Panel 3

Examples of Derivatives:

If $f(x) = x^2$, find $f'(x)$

Means:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$
$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2x\cancel{h} + h^{\cancel{2}} - \cancel{x^2}}{\cancel{h}} =$$
$$= \lim_{h \to 0} 2x + h = 2x$$

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Panel 4

Ex: If $f(x) = \frac{1}{x}$, find equation of tangent line at $x = 2$

Equation of line: $y = mx + b$

a) Slope of tangent at $x = 2$ is $f'(x)$ at $x = 2$ or $f'(2)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} =$$
$$= \lim_{h \to 0} \frac{1}{h} \frac{\cancel{x} - (\cancel{x} + h)}{x(x+h)} = \lim_{h \to 0} \frac{1}{\cancel{h}} \frac{-\cancel{h}}{x(x+h)} = -\frac{1}{x^2}$$

$f'(2) = -1/4$ is slope of line

b) $y = -\frac{1}{4}x + b$ Know at $x = 2$, $f(x) = 1/2 = y$

$\frac{1}{2} = -\frac{1}{4} \cdot 2 + b \Rightarrow b = 1 \Rightarrow \boxed{y = -\frac{1}{4}x + 1}$

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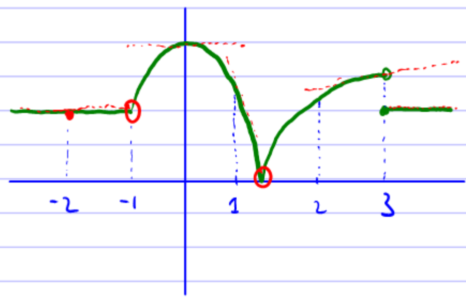
Panel 5

Ex: Consider the following graph of a function f and find the indicated derivatives:



$f'(0) = 0$

$f'(-2) = 0$

$f'(1) < 0$

$f'(2) > 0$

$f'(3)$ d.n.e.

Where is f not continuous? at $x = 3$

Where is f not differentiable? at $x = 3$

and at $x = -1$

$x \approx 1.5$

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Panel 6

Def: f is differentiable at $x = a$ if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

Ex: Is $f(x) = |x|$ differentiable at $x = 0$?

Does $\lim_{h \to 0} \frac{f(\cancel{x}^{0}+h) - f(\cancel{x}^{0})}{h}$ exist for $x = 0$

Check $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \left\{ \begin{array}{l} \lim_{h \to 0^+} \frac{h}{h} = 1 \\ \lim_{h \to 0^-} \frac{-h}{h} = -1 \end{array} \right\}$ d.n.e.

Is $f(x) = |x|$ cont. at $x = 0$?

Check: $\lim_{x \to 0} f(x) = f(0)$ i.e. $\lim_{x \to 0} |x| = 0$

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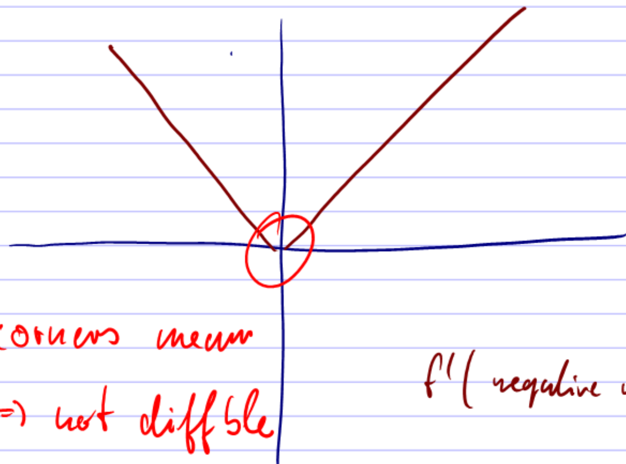
Panel 7

$f(x) = |x|$



corners mean

$\Rightarrow$ not diff'ble

$f'(\text{negative values}) = -1$

$f'(\text{pos. values}) = 1$

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Panel 8

Theorem: If $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$. The converse is

Proof: Know f diff'ble at $x=a$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \text{ exists.}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} =$$

Want: $\lim_{x \to a} f(x) = f(a)$ or

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} (x-a) =$$

$$= f'(a) \cdot \lim_{x \to a} (x-a) = 0$$

q.e.d.

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Panel 9

Consider $f(x) = \begin{cases} x^2+2x & \text{if } x<0 \\ \sin(x) & \text{if } x \geq 0 \end{cases}$

① Is f cont at 0?
② Is f diff'ble at 0?

① $0 = f(0) = \lim_{x \to 0} f(x)$ $\begin{cases} \lim_{x \to 0^+} \sin(x) = 0 \\ \lim_{x \to 0^-} x^2+2x = 0 \end{cases}$ yes, cont. at 0!

② $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} \stackrel{?}{=} \begin{cases} \lim_{h \to 0^+} \frac{\sin(h)}{h} = 1 \\ \lim_{h \to 0^-} \frac{h^2+2h}{h} = 2 \end{cases}$

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Panel 10

Short cuts for differentiation:

$f(x) = x^2 \Rightarrow f'(x) = 2x$

$f(x) = x^3 \qquad f'(x) = 3x^2$

$f(x) = \frac{1}{x} = x^{-1} \qquad f'(x) = -\frac{1}{x^2} = -1 x^{-2}$

$f(x) = \sqrt{x} = x^{1/2} \qquad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}$

Big Q: $f(x) = x^{11} \qquad f'(x) = 11\, x^{10}$

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Panel 11

Power Rule:

If $f(x) = x^p$ then $f'(x) = p\,x^{p-1}$

Ex: $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3}$

$f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

Slope of tangent line to $f(x) = 1/x^3$ at $x = 1$

$f'(x)$ at $x = 1$, $f'(x) = -3x^{-4}$

$f'(1) = \underline{-3}$ is slope of tangent

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Panel 12

Constant Rule

$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$ but $\frac{d}{dx}(c) = 0$

Ex: $f(x) = 5x^2$, $f'(x) = 5 \cdot 2x = 10x$

$f(x) = \frac{5}{9x^{1/3}}$, $f'(x) = \frac{5}{27x^{4/3}}$?

$= \frac{5}{9}x^{-1/3} \xrightarrow{\text{calculus}} f'(x) = \underline{\frac{5}{9}\left(-\frac{1}{3}\right)x^{-4/3}} \xrightarrow{\text{algebra}} -\frac{5}{27x^{4/3}}$

$f(x) = 2 \Rightarrow f'(x) = 0$

$f(x) = 2^3$, $\Rightarrow f'(x) = 0$

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Panel 13

Add / Subtract Rule

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

$$f(x) = x^2 + x^5 - \frac{1}{x} + \pi^3$$

$$f'(x) = 2x + 5x^4 + x^{-2} + 0$$

$$f(x) = x^2(1 - 9x) = x^2 - 9x^3$$

$$x^{1/2}$$

$$\frac{\sqrt{x}}{x} = x^{1/2} x^{-1}$$

$$f'(x) = 2x - 27x^2$$

$$f(x) = \frac{5x^2 - 9\sqrt{x} + 3}{x} = \frac{1}{x}\left(5x^2 - 9x^{1/2} + 3\right) = 5x - 9x^{-1/2} + 3x^{-1}$$

$$f'(x) = 5 + \frac{9}{2}x^{-3/2} - 3x^{-2}$$

$$x^{-1} \cdot 5x^2 = 5x^2 \cdot x^{-1} = 5x$$

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Panel 14

Practice exam

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