Panel 1

Last time:

Shifting, Stretching, Symmetry

$\lim_{x \to a} f(x) = L$

$\lim_{x \to a^+} f(x) = L$

$\lim_{x \to a^-} f(x) = L$

Limit of piecewise defined function

1

Panel 2

$\lim_{x \to 0} \frac{\sin(x)}{x + \tan(x)} = f(x)$ $x = \pm 0.01$, $x = \pm 0.001$

x	y
0.2	0.49
-0.2	0.49

$f(x) = \sin(x)$, $\sin(-x) = -\sin(x)$ odd

$f(-x) = -f(x)$ $\cos(-x) = \cos(x)$ even

$\Rightarrow \tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$

$\Rightarrow \frac{\sin(-x)}{(-x) + \tan(-x)} = \frac{-\sin(x)}{-x - \tan(x)} = \frac{-\sin(x)}{-(x + \tan(x))} = \frac{\sin(x)}{x + \tan(x)}$

$\shortparallel$

$f(-x)$ $= f(x)$

2

Panel 3

Shift + Strech: $c > 0$

①	$f(x)+c$	shetch	Ⓐ
②	$f(cx)$	shrink	Ⓑ
③	$f(x+c)$	move up / ~~down~~	Ⓒ
④	$c\,f(x)$	move right / left	Ⓓ

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Panel 4

18) $\lim\limits_{x \to 0} \dfrac{9^x - 5^x}{x} \left(= \dfrac{0}{0} \right)$ Stuck thus:

x	
0.1	
0.01	
⋮	

14) $\lim\limits_{x \to 0} \dfrac{\overset{a}{\sqrt{x+4}} - \overset{b}{2}}{x} \left(= \dfrac{0}{0} \right) \dfrac{(\overset{a}{\sqrt{x+4}} + \overset{b}{2})}{(\sqrt{x+4} + 2)} = \lim\limits_{x \to 0} \dfrac{\cancel{x}^{1} \cancel{+4} \cancel{-4}}{\cancel{x}(\sqrt{x+4}+2)} = \dfrac{1}{4}$

?) $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x} \left(= \dfrac{0}{0} \right) = \lim\limits_{x \to 0} \dfrac{\cancel{x}(x+3)}{\cancel{x}} = \lim\limits_{x \to 0} x + 3 = 3$

??) $\lim\limits_{x \to 0} \dfrac{x^2 + 9x}{x+3} = 0/3 = 0$

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Panel 5

$$\lim_{x\to -4} \frac{\left(\frac{1}{4}+\frac{1}{x}\right)}{(4+x)} \cdot \frac{4x}{4x} = \lim_{x\to -4} \frac{\cancel{x+4}}{(\cancel{4+x})\,4x} = -\frac{1}{16}$$

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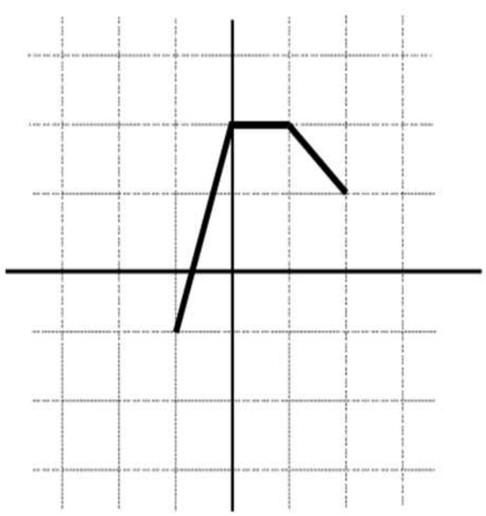
Panel 6

Quiz #1

1 Suppose the graph of a function $f(x)$ looks as indicated in the picture below:



Based on that graph, find the graph of

a) $f(x)-2$;

b) $f(x+1)$

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Panel 7

Quiz #1 - continued

② Compute the following limits:

a) $\lim_{x \to 2} \frac{x^2+1}{x+2}$

b) $\lim_{x \to 2} \frac{x^2-4}{x-2}$

c) $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

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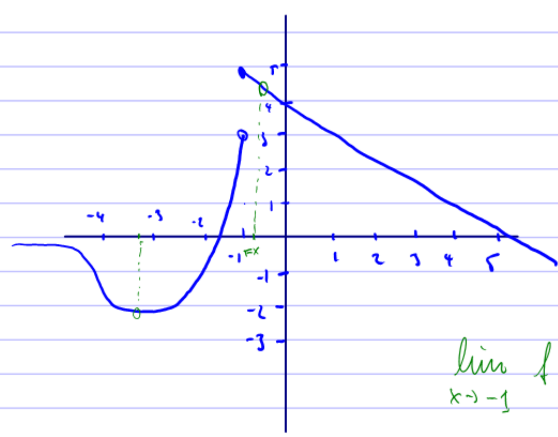
Panel 8

Limits Graphically:

$\lim_{x \to -3} f(x) = -2$

$\lim_{x \to 5.1} f(x) = 0$



$\lim_{x \to -1} f(x) =$ does not exist

a) $\lim_{x \to -1^+} f(x) = 5$

b) $\lim_{x \to -1^-} f(x) = 3$

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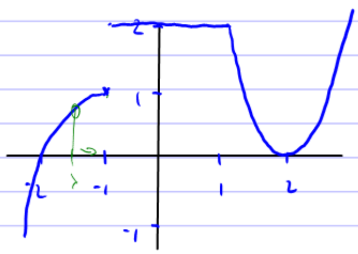
Panel 9

Limits Graphically:



① $\lim_{x \to -2} f(x) = 0$

② $\lim_{\boxed{x \to -1}} f(x) = \text{undefined}$

⑦ $\lim_{x \to -1^-} f(x) = 1$

③ $\lim_{x \to 0} f(x) = 2$

④ $\lim_{x \to 2} f(x) = 0$

⑧ $\lim_{x \to -1^+} f(x) = 2$

⑤ $\lim_{x \to 2^-} f(x) = 0$

⑥ $\lim_{x \to 2^+} f(x) = 0$

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Panel 10

Theorem: $\lim_{x \to a} f(x) = L$

if and only if

$$\underbrace{\lim_{x \to a^+} f(x) = L}_{\text{limit from the right}} \quad \text{and} \quad \underbrace{\lim_{x \to a^-} f(x) = L}_{\text{limit from the left}}$$

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Panel 11

Summary for Limit Calculations

If you can legally substitute the value in question, do it.

you get: $\frac{0}{\#}$ → answer is 0

$\#$ ~ non-zero number

$\frac{\#}{0}$ → not a number, DNE, or ...

$\frac{0}{0}$ → more work

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Panel 12

$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$ ✓

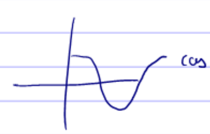
⇓

$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$ ✓ $\qquad \lim_{x \to 4} \frac{x^2 + 16}{x - 4} \qquad \lim_{x \to 4} \frac{x^2 - 16}{x + 4} = 0$

$\lim_{x \to 0} \frac{|x|}{x}$ d.n.e. ↯

$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ $\qquad -x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$

by squeezing theorem $\qquad \Downarrow$ 0



$\lim_{x \to 0} \frac{\sin(7x)}{4x}$ Really Hard:

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Panel 13

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0} \frac{|x|}{x} : \begin{cases} \lim\limits_{x\to 0^+} \frac{|x|}{x} = \lim\limits_{x\to 0^+} \frac{x}{x} = 1 \\ \lim\limits_{x\to 0^-} \frac{|x|}{x} = \lim\limits_{x\to 0^-} \frac{-x}{x} = -1 \end{cases}$$

$\uparrow$

d.n.e.

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Panel 14

Squeezing Theorem

If $f(x)$ is such that $g(x) \leq f(x) \leq h(x)$,
and $\lim\limits_{x\to c} g(x) = \lim\limits_{x\to c} h(x) = L$, then $\lim\limits_{x\to c} f(x) = L$

Remember like this

$$\begin{array}{ccccc} g(x) & \leq & f(x) & \leq & h(x) \\ \downarrow & & \Downarrow & & \downarrow \\ L & & L & & L \end{array}$$

Ex: $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right)$

$$\begin{array}{ccccc} -x & \leq & x\sin\left(\frac{1}{x}\right) & \leq & x \\ \downarrow{\scriptstyle x\to 0} & & \Downarrow & & \swarrow \\ 0 & & 0 & & 0 \end{array}$$

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Panel 15

Believe me: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$\Rightarrow \lim_{x\to 0} \frac{\sin(7x)}{4x} = \lim_{x\to 0} \frac{7}{4}\frac{\sin(7x)}{7x} = \frac{7}{4} \lim_{t\to 0} \frac{\sin(\overbrace{t}^{t=7x})}{t} = \frac{7}{4}\cdot 1$

$\frac{\sin(7x)}{4x} = \frac{\cancel{7}}{4}\frac{\sin(7x)}{\cancel{7}x}$

Want

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Panel 16

Formal Definition of Limit (ε-δ def.)

If for every number $\varepsilon > 0$ there is another number $\delta > 0$ such that

if $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$

then

$\lim_{x\to a} f(x) = L$

Ex: Prove that $\lim_{x\to 3}(4x-5) = 7$

scratch paper

Want: if $|x-3| < \delta$ then $|4x-5-7| < \varepsilon$

if $|x-3| < \delta$ then $|4x-12| < \varepsilon$

if $|x-3| < \delta$ then $4|x-3| < \varepsilon$ or $|x-3| < \varepsilon/4$

Let $\delta = \varepsilon/4$: if $|x-3| < \delta = \varepsilon/4$ then $4|x-3| < 4\frac{\varepsilon}{4} = \varepsilon$

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Panel 17

Prove $\lim_{x \to 3} 4x - 5 = 7$

Let $\delta = \varepsilon/4$. For any number $\varepsilon > 0$ we pick $\delta = \varepsilon/4$. Then: if

$|x-3| < \delta$

$\Rightarrow |x-3| < \varepsilon/4 \Rightarrow 4|x-3| < \varepsilon$

$\Rightarrow |4x-12| < \varepsilon \Rightarrow |4x-5-7| < \varepsilon$

$\Rightarrow |f(x) - 7| < \varepsilon$

q.e.d. quod erat demonstrandum

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Panel 18

Try to prove:

$\lim_{x \to 1} 2x + 3 = 5$ Find $\delta = \ldots$?

$\lim_{x \to 2} x^2 + 1 = 5$ Find $\delta = \ldots$?

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Panel 19

Continuity

The concept of limit is fundamental to Calculus.
All other important concepts are based on it.

Def: A function f is continuous at a number c if $\lim_{x \to c} f(x) = f(c)$

a) $\lim_{x \to c} f(x)$

b) $f(c)$

c) do they agree

Continuity is easy to see if you know the graph

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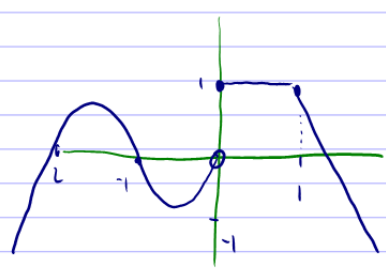
Panel 20

Continuity Graphically



continuity = no gap, no holes

technically:

Is f cont at 1? $\lim_{x \to 1} f(x) = 1$, $f(1) = 1$

Is f cont. at 0? $\lim_{x \to 0} f(x) = \text{d.n.e.}$

Not

$$\left.\begin{array}{l} \lim_{x \to 0^+} f(x) = 1 \\ \lim_{x \to 0^-} f(x) = 0 \end{array}\right\} \text{different}$$

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Panel 21

Continuity and Piecewise-defined Functions

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 42 & x = 3 \end{cases}$$

Is it cont. at $x=3$?

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = 6$$

not cont.

$$f(3) = 42$$

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

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Panel 22

Let

$$f(x) = \begin{cases} kx & \text{if } x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$$

For what value of k is $f(x)$ cont. at $x = 1$.

By def: f continuous at $x=1$ means

$$\lim_{x \to 1} f(x) = f(1) \quad \stackrel{?}{=} \quad f(1) = 3$$

$$\lim_{x \to 1} f(x) = 3 \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2+2 = 3$$

my choice!

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} kx = k = 3$$

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Panel 23

Limits involving Infinity

Next time

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Panel 24

Theorem for Cont. Functions

If f, g are two continuous functions, then

$f+g$ cont.

$f-g$ cont.

$f\cdot g$ cont.

f/g cont. if $g(x)\neq 0$

$f\circ g$ cont if defined

Ex: $f(x)=\dfrac{x^2-9}{x-17}$ is cont. for all $x\neq 17$

For HW, check web pages, see you Wed!

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