Panel 1

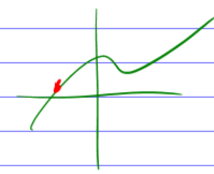
Different Terms for "Derivative":

Definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Rules: Power Rule, $\frac{d}{dx} e^x = e^x$, $\frac{d}{dx} \ln(x) = \frac{1}{x}$

Rate of change

Marginal Revenue, Cost, Profit



Slope of tangent

Physics: velocity

decreasing, increasing

1

Panel 2

Derivatives of special functions

If $f(x) = e^x$ then $f'(x) = e^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

Power Rule does not apply!!!

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right) = \boxed{e^x}$$

h	$\frac{e^h - 1}{h}$
0.0001	1.00005000...
⋮	⋮
	1

2

Panel 3

Derivative of e^x and $\ln(x)$:

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln(x) = 1/x$$

Ex: $f(x) = e^2 + e^x + x^2 + x^e + \ln(x) + \frac{1}{x} + x + 1$ ($\frac{1}{x} = x^{-1}$)

$f'(x) = 0 + e^x + 2x + ex^{e-1} + \frac{1}{x} - x^{-2} + 1 + 0$

3

Panel 4

Ex: $C(x) = 5\cdot\ln(x) + 7e^x - 5x^4 + 2\sqrt{x} + \underbrace{\ln(\pi) + \sqrt{2} + e^7 + 9}$ ($\sqrt{x} = x^{1/2}$)

Find the rate of change of the marginal cost function.

① Marginal cost. $C'(x) = 5\cdot\frac{1}{x} + 7e^x - 20x^3 + 2\cdot\frac{1}{2}x^{-1/2}$ ($\frac{1}{x} = x^{-1}$) $\boxed{\text{say } C'(10) > 0}$

Rate of change of $C'(x) = C''(x) = -5x^{-2} + 7e^x - 60x^2 - \frac{1}{2}x^{-3/2}$

Recall: $\frac{d}{dx} 5x^3 = 15x^2$

$\frac{d}{dx} 7\sqrt{x} = \frac{d}{dx} 7x^{1/2} = 7\cdot\frac{1}{2}x^{-1/2}$

$\frac{d}{dx}\frac{9}{x^3} = \frac{d}{dx} 9x^{-3} = 9\cdot(-3)x^{-4}$

$\frac{d}{dx} 5\sqrt[3]{x^2} = 5\left(x^2\right)^{1/3}$

$\frac{d}{dx}\left(5x^{2/3}\right)$

$5\cdot\frac{2}{3}x^{-1/3}$

4

Panel 5

Higher Order Derivatives

If $f(x)$ is a (differentiable) function, then $f'(x)$ is again a function.

$f(x) \Rightarrow f'(x)$ is 1^{st}-derivative $= \frac{d}{dx} f$

$f'(x) \Rightarrow f''(x)$ is 2^{nd}-deriv. $= \frac{d^2}{dx^2} f$

$f''(x) \Rightarrow f'''(x)$ is 3^{rd} deriv $= \frac{d^3}{dx^3} f$ ...

$\vdots$

$f^{(n)}(x)$ is n-th derivative

or $\frac{d^n}{dx^n} f = f^{(n)}(x)$

5

Panel 6

Ex: If $f(x) = 3x^4 - 9x^2 + 5x$, find f''

$f'(x) = 12x^3 - 18x + 5$

$f''(x) = 36x^2 - 18$

Ex: $f(x) = 5e^x - 7\ln(x) + \sqrt{x}$. Find f'''

$\sqrt{x} = x^{1/2}$, $-7x^{-1}$

$f'(x) = 5e^x - 7/x + \frac{1}{2}x^{-1/2}$

$f''(x) = 5e^x + 7x^{-2} - \frac{1}{4}x^{-3/2}$

$f'''(x) = 5e^x - 14x^{-3} + \frac{1}{4}\cdot\frac{3}{2}\cdot x^{-5/2}$

6

Panel 7

Some times higher order derivatives become easy:

Ex: $f(x) = 3x^3 - 2x^2 + x$. Then

$f'(x) = 9x^2 - 4x + 1$

$f''(x) = 18x - 4$

$f'''(x) = 18$

$f''''(x) = f^{(4)}(x) = 0$

$f'''''(x) = f^{(5)}(x) = 0$

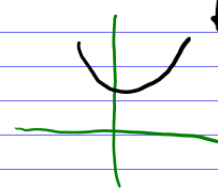
$\vdots$

$f^{(100)}(x) = 0$

7

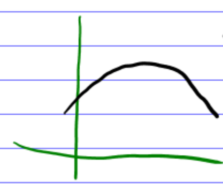
Panel 8

Meaning of f''



$f'' > 0 \Rightarrow f'$ (goes up) increasing

$f'' < 0 \Rightarrow f'$ (goes down) decreasing



$f'' = 0 \Rightarrow$ possible inflection points

Ex: $f(x) = 2x^3 - 6x^2$ investigate concavity. Where is it concave up? $(1, \infty)$

$f'(x) = 6x^2 - 12x$

$f''(x) = 12x - 12 = 0$

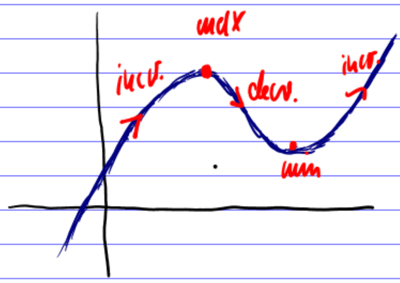
$x = 1$ possible infl. point.

	0	1	2
f''	$-$		$+$
f	$\cap$		$\cup$

8

Panel 9

More appl. of derivative: Max / Min of a Function



f' is positive.

$\Rightarrow$ f is increasing

f' is negative

$\Rightarrow$ f is decreasing

Theorem: If f has max or min at $x=c$,
then $f'(c)=0$ or $f'(c)$ is undefined.

Def: critical points are where $f'(x)=0$ or
$f'(x)$ is undefined.

9

Panel 10

How to Find local Max / Min

either max or min

Ex: $f(x) = 2x^3 + 3x^2 - 12x - 3$ Find all local extrema

① $f'(x) = 6x^2 + 6x - 12$

② $0 = 6x^2 + 6x - 12$
$= 6(x^2 + x - 2) = 6(x+2)(x-1)$, $x = -2, 1$ critical

③

	-3	-2	0	1	Billions
f'	+		−		+
f	↗		↘		↗

$x=-2$ max

$x=1$ min

Recipe

① $f'(x)$

② $f'(x)=0$ or undefined
(critical points)

③ Make a table of signs of f'
between all critical points.

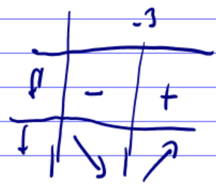
④ Read off the answers

10

Panel 11

Ex: $f(x) = x^2 + 6x - 8$ Find local max/min

$f'(x) = 2x + 6 = 0 \Rightarrow x = -3$ critical point.



$x = -3$ is a min.

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b = 0$

$\boxed{x = -b/2a}$

11

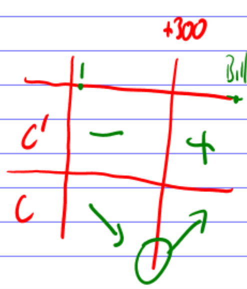
Panel 12

Ex: Suppose $C(x) = \frac{360\,000}{x} + 4x$ is a cost function based on the inventory $x > 0$. How much inventory should you carry to minimize the cost?

$C'(x) = -360000x^{-2} + 4 = -\frac{360\,000}{x^2} + 4 = 0$

$4 = \frac{360000}{x^2} \Leftrightarrow x^2 = \frac{360000}{4} = 90000$

$x = \pm 300$



$x = 300$ gives min. cost, and that min cost is

$C(300) = \frac{360000}{300} + 4 \cdot 300$

12