Math 1303: Practice Exam 3

1. What is the meaning of the following symbols:

    a) $\int f(x)\, dx$

indefinite integral or antiderivative

    b) $\int_a^b f(x)\, dx$

Definite integral or area under a curve (if f(x) >= 0)

2. Find the following indefinite integrals
    a) $\int 9x^2 + 4\, dx$

$$\int \left(9x^2 + 4x\right) dx = 3x^3 + 2x^2 + \text{constant}$$

    b) $\int 3x^8 + 6e^x + \frac{7}{x} - \frac{8}{x^3} - 4\sqrt[3]{x^2}\, dx$

$$\int \left(3x^8 + 6e^x + \frac{7}{x} - \frac{8}{x^3} - 4x^{2/3}\right) dx =$$
$$-\frac{12 x^{5/3}}{5} + \frac{x^9}{3} + \frac{4}{x^2} + 6 e^x + 7 \log(x) + \text{constant}$$

    c) $\int (e^x - x^2 - 5)dx$

$$\int \left(e^x - x^2 - 5\right) dx = -\frac{x^3}{3} - 5x + e^x + \text{constant}$$

    d) $\int (\frac{6}{x} + \frac{4}{x^4} - \sqrt{x})dx$

$$\int \left(\frac{6}{x} + \frac{4}{x^4} - \sqrt{x}\right) dx = -\frac{2\left(x^{9/2} - 9 x^3 \log(x) + 2\right)}{3 x^3} + \text{constant}$$

(this is a heavily simplified answer. The unsimplified answer would also work)

    e) $f(x)$ if $f'(x) = \sqrt{x} - 3$ and $f(4) = -1$.

$$\int \left(\sqrt{x} - 3\right) dx = \frac{2 x^{3/2}}{3} - 3x + \text{constant}$$

First we integrate to find f(x) = _____. Then we use the fact that f(4) = -1 to find C, because f(4) = 2/3 * 8 – 3*4 + C = -1, so that C = 17/3

3. Evaluate the following definite integrals and simplify your answer
    a) $\int_{-1}^{2} 10x^4 - x\, dx$

$$\int_{-1}^{2} \left(10 x^4 - x\right) dx = \frac{129}{2}$$

b) $\int_1^e x - \frac{1}{x} dx$

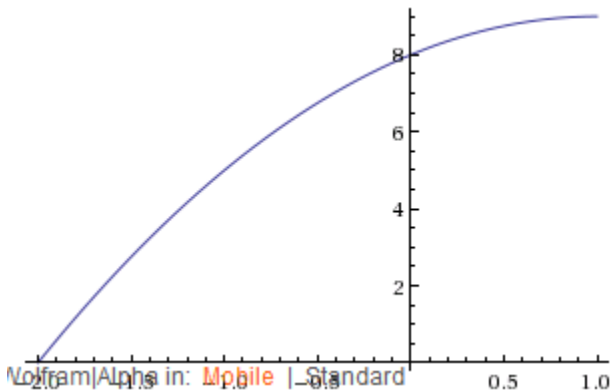$\int_1^e \left(x - \frac{1}{x} \right) dx = \frac{1}{2} \left(e^2 - 3 \right) \approx 2.19453$

f) $\int_{-1}^1 (t^2 - t^4) dt$

$\int_{-1}^1 \left(t^2 - t^4 \right) dt = \frac{4}{15} \approx 0.266667$
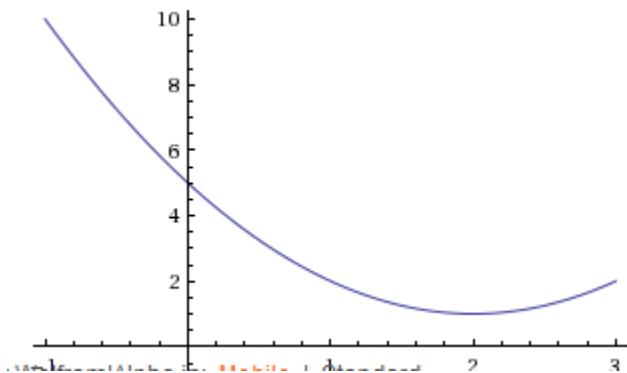
g) $\int_1^e (x^2 - \frac{1}{x}) dx$

$\int_1^e \left(x^2 - \frac{1}{x} \right) dx = \frac{1}{3} \left(e^3 - 4 \right) \approx 5.36185$

4. Find the area under the graph $y = 2x - x^2 + 8$ from $x = -2$ to $x = 1$. Make sure to **sketch the function and shade** the region.
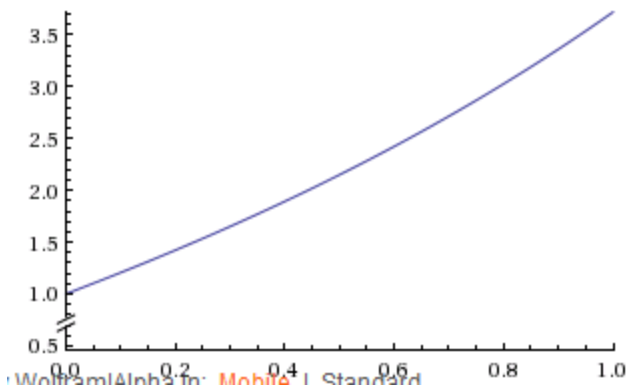
Area = $\int_{-2}^1 \left(2x - x^2 + 8 \right) dx = 18$

(b) Find the area under the curve $y = x^2 - 4x + 5$ from x=-1 to x=3. Sketch the region.
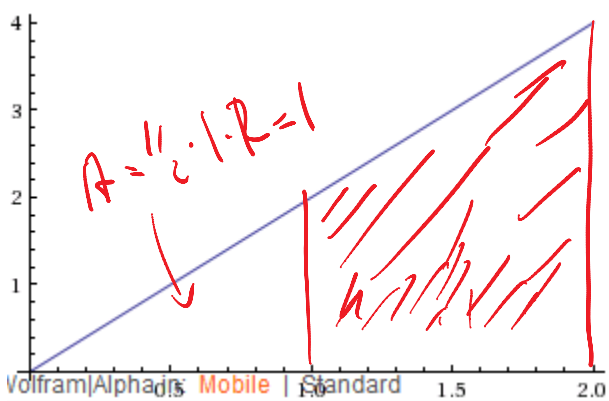

Area = $\int_{-1}^3 \left(x^2 - 4x + 5 \right) dx = \frac{40}{3}$

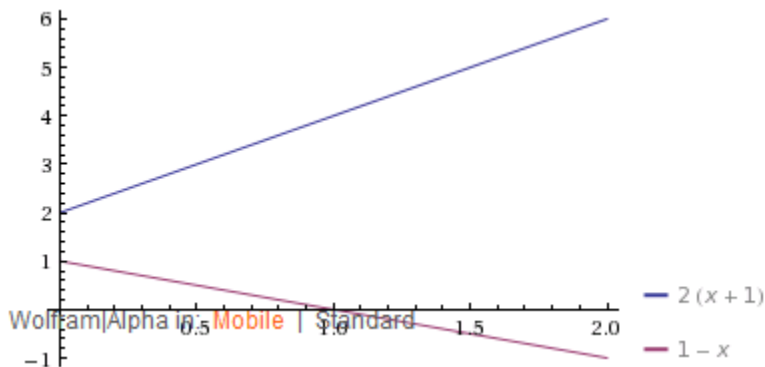(c) Find the area under the curve $y = e^x + x$ from $x = 0$ to $x = 1$

$$\int_0^1 (e^x + x)\,dx = e - \frac{1}{2} \approx 2.21828$$

Area = _____

(d) Find the area under the curve $y = 2x$ as $x = 1$ to $x = 2$ geometrically, then verify via integration.

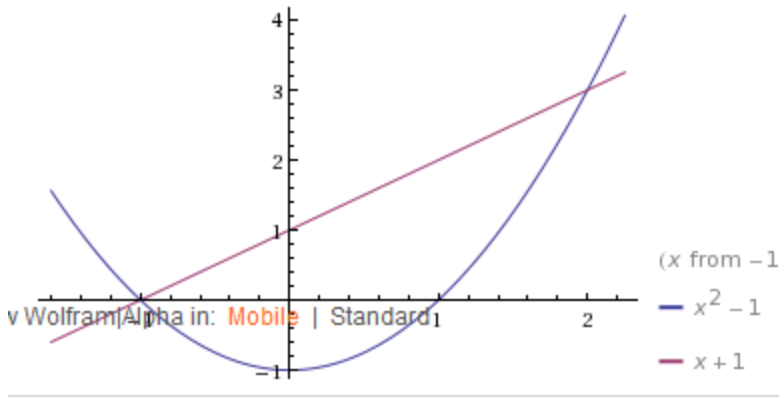$A = \frac{1}{2} \cdot 1 \cdot R = 1$

big triangle area $= \frac{1}{2} \cdot 2 \cdot 4 = 4$

shaded part $= 4 - 1 = 3$

Integrate $\int_1^2 2x\,dx = x^2 \big|_1^2 = 4 - 1 = 3$

5. Find the area between $y = 2x + 2$ and $y = 1 - x$ from $x = 0$ to $x = 2$.

— $2(x+1)$

— $1-x$

$$\int_0^2 ((2x+2) - (1-x))\,dx = 8$$

Area:

(b) Find the area between $f(x) = x^2 - 1$ and $g(x) = x + 1$. Make sure to **sketch the functions and shade the region.**

(x from −1

— $x^2 - 1$

— $x + 1$

Area = $\int_{-1}^{2} \left((x+1) - (x^2 - 1) \right) dx = \dfrac{9}{2}$

6. Suppose the marginal cost for producing x number of widgets is $C' = 3e^x - 9q^2 - 300$, and the fixed cost is 103. Find the formula for the cost function.

C(q) = 3e^q – 3q^3 – 300q + C, C(0) = 103 (fixed cost), so 103 = 3e^0 – 0 – 0 + C, so that C = 100

C(q) = 3e^q – 3q^3 – 300q + 100

7. After t hours of operation, a coal mine is producing coal at a rate of $40 + 2t - 9t^2$ tons of coal per hour. Find the formula for the output of the coal mine after t hours of operation if we know that after 2 hours, 80 tons have been mined.

Output(t) = 40t + t^2 – 3t^3 + C. We also know that Output(2) = 80, so that

40*2 + 2^2 – 3*2^3 + C = 80 so that

60 + C = 80 so that C = 20.

Thus: Output(t) = 40t + t^2 – 3t^3 + 20

8. How long will it take for $ 100 to amount to $ 1000 if invested at 6% compounded monthly? Express the answer in years, rounded to two decimal places.

solve $1000 = 100 \left(1 + \dfrac{0.06}{12}\right)^{12t}$

t = ln(10)/(12*ln(1.005)) = 38.47 years

9. An investor has a choice of investing a sum of money at 8% compounded semiannually or at 7.8% compounded monthly. Which is the better of the two rates?

Effective rate of 8.0% compounded semiannually is (1 + 0.08/2)^2 – 1 = 0.0816 or 8.16%

Effective rate of 7.8% compounded monthly is (1 + 0.078/12)^12 – 1 = 0.08085 or 8.08%

Thus, the first rate is better.

9. Suppose you invest $250 at a nominal interest rate of 7% compounded quarterly.
   a) What is the *effective* rate of interest?
   b) How much is your investment worth after 5 years?
   c) How would you use the *effective* rate to compute the answer for part (b)

a) The effective rate is (1 + 0.07/4)^4 – 1 = 0.071859 or about 7.2%
b) S = 250 * (1 + 0.07/4)^4*5 = $353.69
c) S = 250 * (1 + 0.071859)^5 = $353.69

10. Suppose $8,000 is invested in an account. How much money is in the account in 6 years if the interest rate is 5% compounded: a) monthly   b) continuously?

a) 8000*(1+0.05/12)^(6*12) = $10792.14

b) 8000*e^(0.05*6) = $10798.87 (slightly more that compounding 12 times only)

11. Suppose $700 amounted to $801.06 in a savings account after two years. If interest was compounded quarterly, find the *nominal* rate of interest.

$$\text{solve} \quad 801.06 = 700\left(1 + \frac{r}{4}\right)^{4 \times 2} \quad \text{for} \quad r$$

$$r = \left(\sqrt[8]{\frac{801.06}{700}} - 1\right)4$$
= 0.068 or 6.8%

12. A bank account pays 5.3% annual interest, compounded monthly. How much must be deposited now so that the account contains exactly $ 12,000 at the end of one year?

=PV(0.053/12,12,0,12000) = $11,381.89

13. A trust fund for a **10**-year old child is being set up with a single payment so that at age **21** the child will receive$37,000. What is that payment if an interest rate of 6% is compounded monthly *(if you use Excel for this problem, write down the exact formula you use to obtain your answer)*?

=PV(0.06/12,12*11,0,37000 = $19,154.97

14. Suppose $ 50 is placed in a savings account each month for four years. If no further deposits are made, (a) how much is in the account after six years, and (b) how much of this amount is compound interest? Assume that the savings account pays 6% compounded monthly *(if you use Excel for this problem, write down the exact formula you use to obtain your answer).*

=FV(0.06/12,4*12,-50,0) = $2,704.89

For (b), we deposit $50 12*4=48 times, so we deposit $50 * 48 = $2,400, so the difference of $304.89 is the amount of compound interest collected.

16. Sylvia inherited $30,000. She wants to buy a car, but also wants to put money away for graduate school. She knows she will require $26,000 in four years for graduate school.
    a) How much money should she invest **now** so she will have $26,000 in 4 years if the bank offers an annual interest rate of 6.5%, compounded monthly?
    b) How much money does Sylvia have left to buy her car?

=PV(0.065/12,12*4,0,26000 = $20,061.41, which will yield $26,000 in 4 years.
She therefore has $30,000 - $20.061.41 = $9,938.59 left to purchase a car now.

17. Consider the graph of a function *f(x)* shown below and determine whether the given quantity is approximately positive, negative, or zero:
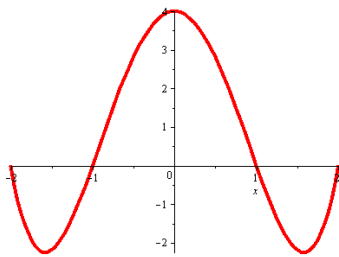
a) $\int_{-1}^{1} f(x)\,dx > 0$ *f(0)>0*   *f '(0)=0*   *f ''(0)<0*

b) $\int_{1}^{2} f(x)\,dx < 0$ *f(1)=0*   *f '(1)<0*   *f ''(1)=0*

c) $\int_{-1}^{2} f(x)\,dx > 0$ *f(-1)=0*   *f '(-1)>0*   *f ''(-1)=0*

For the following questions, you will *have* to use Wolfram Alpha and/or Excel. Please write down your answers together with the formula or steps you used to produce your answer.

A. Evaluate the following expressions: $\int \dfrac{2x^2}{5-x^3}\,dx$ $\int_{0}^{1} \dfrac{32x^2}{(5-x)^4}\,dx$

integrate 2x^2 / (5-x^3)

$\int \dfrac{2x^2}{5-x^3}\,dx = -\dfrac{2}{3}\log(x^3-5) + \text{constant}$

integrate 32x^2 / (5-x)^4 from 0 to 1

$\int_{0}^{1} \dfrac{32x^2}{(5-x)^4}\,dx = \dfrac{1}{30} \approx 0.0333333$

B. A student is saving for a vacation she is planning to take after graduation in *three* years. How much will she have saved if she deposits $40 each month into an account that pays 4% compounded monthly?

=FV(0.04/12,3*12,-40,0) = $1,527.26

(To estimate: she makes 12*3 = 36 payments of $40 each, which amounts to $1,440, plus the compound interest)

C. A parent wants her child to have $40,000 for College in 10 years. Her bank offers 4% interest, compounded monthly, and she can afford monthly payments of $250. What will she need to do to achieve her goal?

=PV(0.04/12,12*10,-250,40000) = -$2,138.10, i.e. the parent should put up a lump sum of $2,138.10 plus $250 per month.
(To estimate: $250 per month for 10 years amount to $250 * 12 * 10 = $30,000, not including any compound interest. Thus, the initial investment should be less than $10,000, so our answer makes sense)