Panel 1

Last Time

Continuity: f is cont. at $x=a$: ① $f(a)$ ② $\lim_{x\to a} f(x)$ ③ Do they agree?

geometrically: no hole, gaps

The derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

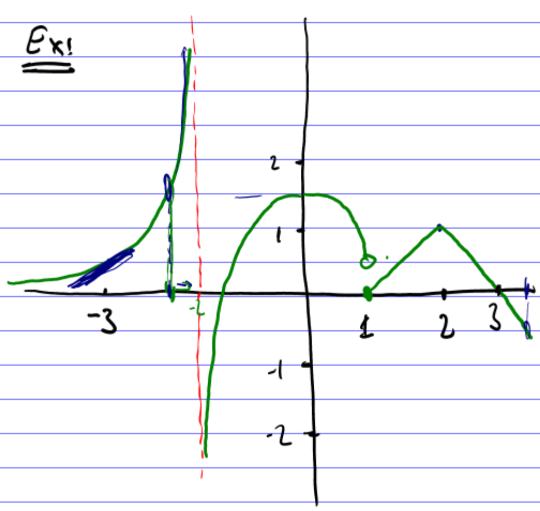
geometrically: slope of tangent line

Notation $f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx} = \frac{dy}{dx} = y'$

1

Panel 2

Ex:



a) $\lim_{x\to 3} f(x) = 0$

b) $\lim_{x\to 1^+} f(x) = 0$

c) $\lim_{x\to -2^-} f(x) = \infty$ (undef.)

d) $\lim_{x\to -\infty} f(x) = 0$

e) $\lim_{x\to 0} f(x) \approx 1\frac{1}{2}$

g) $\lim_{x\to 1} f(x) =$ undef

h) $f'(-3)$ pos

i) $f'(0)$ zero

j) $f'(3)$ neg

k) $f'(2)$ undef

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Panel 3

Ex: $f(x) = \begin{cases} \frac{x^2-9}{x^2-x-6} & \text{if } x \neq 3 \\ K & \text{if } x = 3 \end{cases}$

What should k be, if anything, so that f(x) is continuous at x = 3? Pick $k = 6/5$

① $f(3) = K$ ✓

② $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = 6/5$ ✓

③ $K = 6/5$ ✓

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Panel 4

Back to Derivatives:

Ex: $f(x) = 3x^2$. Find $f'(x) =$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} =$$

$$= \lim \frac{3(x^2+2xh+h^2) - 3x^2}{h} =$$

$$= \lim \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} = \lim \frac{\cancel{h}(6x+3h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h = 6x$$

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Panel 5

Find $\frac{d}{dx} f$, where $f(x) = \frac{1}{x}$ $= x^{-1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = -\frac{1}{x^2} = -x^{-2}$$

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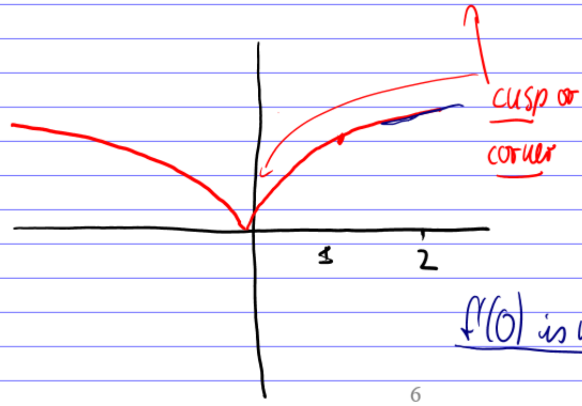
Panel 6

Derivatives are defined via limits. Thus, they do not necessarily have to exist:

Def: If the graph of a function does not have a unique tangent line at a point, it is not differentiable at that point.

Ex:



a) $f'(2) > 0$

b) $f'(1) > 0$

$f'(0)$ is undefined

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Panel 7

Derivatives as limits can get complicated ⇒ need short cut:

Ex: $f(x) = x^1 \Rightarrow f'(x) = 1x^0$

$\lim_{h \to 0} \frac{x+h-x}{h}$

$f(x) = x^2 \Rightarrow f'(x) = 2x^1$

lim ...

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = x^4 \Rightarrow f'(x) = 4x^3$

$f(x) = x^5 \quad f'(x) = 5x^4$

Power Rule

$f(x) = x^n \quad f'(x) = n x^{n-1}$

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Panel 8

Rules for Differentiation - Part 1

The Power Rule:

$$\frac{d}{dx} x^n = n x^{n-1}$$

The Constant Rule:

$$\frac{d}{dx} c = 0$$

The Constant Factor Rule:

$$\frac{d}{dx} (c \cdot f(x)) = c f'(x)$$

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Panel 9

Examples: $f(x) = 3$ $\quad f'(x) = 0$

$g(x) = x^2$ $\quad f'(x) = 2x$

$h(x) = 3x^5$ $\quad h'(x) = 3 \cdot 5x^4 = 15x^4$

$k(x) = \sqrt{x} = x^{1/2}$ $\quad k'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\ell(x) = \frac{9}{5}\sqrt[3]{x} = \frac{9}{5}x^{1/3}$, $\ell'(x) = \frac{9}{5}\frac{1}{3}x^{-2/3} = \frac{3}{5}x^{-2/3}$

$m(x) = \frac{4}{\sqrt[3]{x^2}} = 4x^{-2/3}$, $m'(x) = 4\left(-\frac{2}{3}\right)x^{-5/3}$

$n(x) = 3\pi e^2$, $n'(x) = 0$

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Panel 10

More (simple) Differentiation Rules

The Sum/Difference Rule: $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$

Ex: $f(x) = x^2 + 3x^1 - 7$

$f'(x) = 2x + 3 - 0$

$g(x) = 5x^2 - \frac{7}{x^2} + 9\sqrt[3]{x^4} + \pi^2$

$5x^2 - 7x^{-2} + 9x^{4/3} + \pi^2$

$g'(x) = 10x + 14x^{-3} + 9 \cdot \frac{4}{3}x^{1/3} + 0$ ✓

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Panel 11

More Examples:

A) $f(x) = x^5$

B) $f(x) = x^3 + 9$

C) $f(x) = 3x^2 - 7x$

D) $f(x) = 1/x^3 = x^{-3}$

E) $f(x) = 3x^2 - \frac{7}{x} + \sqrt{x} =$

$f'(x) = 6x + 7x^{-2} + \frac{1}{2}x^{-1/2}$

F) $f(x) = 9x^3 - 8\underbrace{\sqrt{x^3}}_{x^{3/2}} + 5\underbrace{\sqrt[3]{x^2}}_{x^{2/3}} + \pi$

$f'(x) = 27x^2 - 8\cdot\frac{3}{2}x^{1/2} + 5\cdot\frac{2}{3}x^{-1/3}$

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Panel 12

Ex: $f(x) = x^4(3x^2 - 2x + 1) = 3x^6 - 2x^5 + x^4$

$f'(x) = 18x^5 - 10x^4 + 4x^3$

$g(x) = (2x-3)^2 = 4x^2 - 12x + 9$

$\boxed{g'(x) = 8x - 12}$

$h(x) = (x - \frac{1}{x})^2$

HW

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Panel 13

More complicated differentiation Rules

Product Rule: $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))\cdot g'(x)$

or use Wolfram Alpha

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Panel 14

Wolfram Alpha:

derivative of (2x-3)^2

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Panel 15

Why study Derivatives

$$\frac{f(x+h)-f(x)}{h} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

or rate of change

$$\frac{f(x+h)-f(x)}{h} = \text{average rate of change}$$

$$f'(x) = \text{instantaneous rate of change}$$

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Panel 16

Ex: Suppose the position function of an object is given by $f(t) = 3t^2 + 5$

a) Find avg. rate of change over [2,3]

$$\frac{f(3)-f(2)}{3-2} = \frac{32-17}{1} = 15 \text{ mph} = \frac{\text{dist}}{\text{time}}$$

b) how fast is the car going when $t=2$?

$f'(t) = 6t \Rightarrow f'(2) = 12$ or at $t=3$ it is $f'(3) = 18$

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Panel 17

Ex: Let $p = 100 - q^2$ be a demand function.
How fast is price changing when $q = 5$?

$p'(q) = -2q$

$p'(5) = -10$ (i.e. if you rev'd up production, the price would go down!

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Panel 18

Ex: Suppose revenue function is $R(q) = 9q - q^3$ and the current level of production is at $q = 2$ (thousand). Should you increase or decrease production?

Want: rate of change in Revenue

$\boxed{\text{quiz on Wed.}}$

$R'(q) = 9 - 3q^2$, so $R(2) = \underline{\underline{-3}}$

Please decrease production!

What level of production gives max revenue!?! $\boxed{\text{next time}}$

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Panel 19

476 #30

$$f(x) = \begin{cases} 3x+5 & \text{if } x \geq -2 \\ 2 & \text{if } x < -2 \end{cases}$$

$f(-2) = -1$

If f cont. at -2 ?

No!

$\lim_{x \to -2} f(x) = \text{undef}$

But f is cont. everywhere else!

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