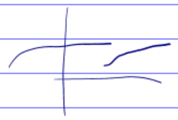
Panel 1

## Last Time

**Continuity:** $f$ is cont. at $x = a$ : (1) $f(a)$ is defined

(2) $\lim_{x \to a} f(x)$ is defined

(3) They agree



graphically: no holes/gaps

**The derivative** $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

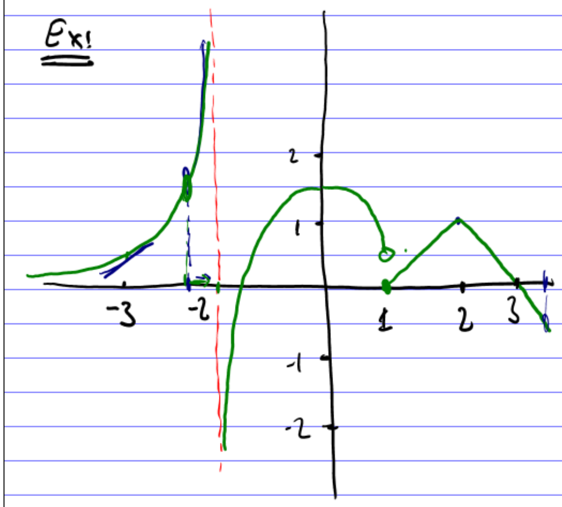
prime

**Notation:** $f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx}$

graphically: slope of tangent

Panel 2

**Ex:**



a) $\lim_{x \to 3} f(x) = 0$

b) $\lim_{x \to 1^+} f(x) = 0$

c) $\lim_{x \to -2^-} f(x) = \infty$ (undef.)

d) $\lim_{x \to -\infty} f(x) = 0$

e) $\lim_{x \to 0} f(x) = 1\frac{1}{2}$

g) $\lim_{x \to 1} f(x) =$ undef

h) $f'(-3)$ pos

j) $f'(3)$ neg

i) $f'(0)$ zero

k) $f'(2)$ undef.

Panel 3

Ex:
$$f(x) = \begin{cases} \frac{x^2-9}{x^2-x-6} & \text{if } x \neq 3 \\ K & \text{if } x = 3 \end{cases}$$

What should k be, if anything, so that f(x) is continuous at $x = 3$? $k = 6/5$

① $f(3) = K$

② $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x^2-x-6} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \lim_{x \to 3} \frac{x+3}{x+2} = \frac{6}{5}$

③ The ① = ② $\Rightarrow$ Yes, if $k = 6/5$

Panel 4

## Back to Derivatives:

Ex: $f(x) = 3x^2$. Find $f'(x)$ (using the definition)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} =$$

$$= \lim_{h \to 0} \frac{3(x^2+2xh+h^2) - 3x^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} = \lim_{h \to 0} \frac{h(6x+3h)}{h} = \lim_{h \to 0} 6x + 3h =$$

$$= 6x$$

Panel 5

Find $\frac{d}{dx} f$, where $\boxed{f(x) = \frac{1}{x}} = x^{-1}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{\boxed{h}} =$$

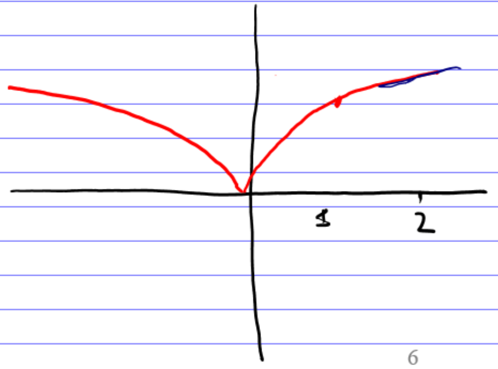
$$= \lim_{h\to 0} \frac{-\cancel{h}}{(x+h)x} \cdot \frac{1}{\cancel{h}} = \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2} = -x^{-2}$$

Panel 6

Derivatives are defined via limits. Thus, they do not necessarily have to exist:

**Def:** If the graph of a function does not have a unique tangent line at a point, it is not differentiable at that point. (i.e. if it has a cusp or corner

**Ex:**



a) $f'(2)>0$

b) $f'(1)>0$

$f'(0)$ is undefined

Panel 7

Derivatives as limits can get complicated ⇒ need short cut:

**Ex:** $f(x) = x^1 \Rightarrow f'(x) = 1x^0$

$\lim_{h \to 0} \frac{x+h-x}{h} = 1$

$f(x) = x^2 \Rightarrow f'(x) = 2x^1$

$\lim \ldots = 2x$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = x^{4} \Rightarrow f'(x) = 4x^{3}$

$f(x) = x^5 \Rightarrow f'(x) = 5x^4$

$f(x) = x^{n} \Rightarrow \boxed{f'(x) = n \cdot x^{n-1}}$ Power Rule

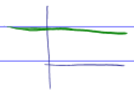
Panel 8

## Rules for Differentiation - Part I

**The Power Rule:**

$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ or $\boxed{\frac{d}{dx} x^n = nx^{n-1}}$

**The Constant Rule:** $f(x) = c \Rightarrow f'(x) = 0$ or $\boxed{\frac{d}{dx} c = 0}$



**The Constant Factor Rule:** $f(x) = c\,g(x) \Rightarrow f'(x) = c\,g'(x)$

or $\boxed{\frac{d}{dx} c\,g(x) = c\,g'(x)}$

Panel 9

**Examples:** $\underline{f(x) = 3}$ $f'(x) = 0$

$g(x) = x^2$ $g'(x) = 2x^1 = 2x$

$h(x) = 3x^5$ $h'(x) = 3 \cdot 5x^4 = 15x^4$

$k(x) = \sqrt{x} = x^{1/2}$ $k'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$l(x) = \frac{9}{5}\sqrt[3]{x} = \frac{9}{5}x^{1/3} \Rightarrow l'(x) = \frac{9}{5} \cdot \frac{1}{\cancel{3}}x^{-2/3} = \underline{\underline{\frac{3}{5}x^{-2/3}}}$

$m(x) = \frac{4}{\sqrt[3]{x^2}} = 4x^{-2/3} \Rightarrow m'(x) = 4\left(-\frac{2}{3}\right)x^{-2/3-1} = \underline{\underline{4\left(-\frac{2}{3}\right)x^{-5/3}}}$

$n(x) = 3\pi e^2$ $\underline{n'(x) = 0}$

Panel 10

## More (simple) Differentiation Rules

**The Sum/Difference Rule:** $\boxed{\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)}$

**Ex:** $f(x) = x^2 + 3x^1 - 7$

$\underline{f'(x) = 2x + 3 - 0}$

$g(x) = 5x^2 - \frac{7}{x^2} + 9\sqrt[3]{x^4} + \pi^2$ $\quad x^{4/3}$

$\boxed{g'(x) = 10x + 7 \cdot 2x^{-3} + 9\frac{4}{3}x^{1/3} + 0}$

Panel 11

## More Examples:

A) $f(x) = x^5$

B) $f(x) = x^3 + 9$

C) $f(x) = 3x^2 - 7x$

D) $f(x) = 1/x^3 = x^{-3}$ $f(x) = -3x^{-4}$

$= 6x + 7x^{-2} + \frac{1}{2}x^{-1/2}$

E) $f(x) = 3x^2 - \frac{7}{x} + \sqrt{x} =$ $f'(x) = 6x + 7x^{-2} + \frac{1}{2}x^{1/2 - 1}$

$\underline{3x^2} - 7x^{-1} + x^{1/2}$

F) $f(x) = \underline{9x^3} - 8\sqrt{x^3} + 5\sqrt[3]{x^2} + \pi$

$-8\ x^{3/2}$ $5 \cdot x^{2/3}$

$f'(x) = 27x^2 - 8\left(\frac{3}{2}\right)x^{\frac{3}{2}-1} + 5\ \frac{2}{3}\ x^{2/3 - 1}$ = <u>simplify</u>

Panel 12

**Ex:** $f(x) = x^4(3x^2 - 2x + 1)$ = foil then differentiate

$g(x) = (2x - 3)^2 = (2x-3)(2x-3) = 4x^2 - 12x + 9$

$g'(x) = 8x - 12$

$h(x) = \left(x - \frac{1}{x}\right)^2$ HW

Panel 13

## More complicated differentiation Rules

**Product Rule:** $\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ messy

**Quotient Rule:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) - f(x)g'(x)}{[g(x)]^2}$ very messy!

**Chain Rule:** $\frac{d}{dx} f(g(x)) = f'(g(x))\cdot g'(x)$ !

not bad!

or use...

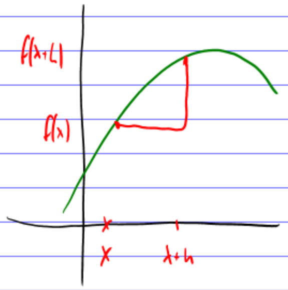
Panel 14

Wolfram Alpha

type: derivative of (2x-3)^2 ↵

Panel 15

# Why study Derivatives



$$\frac{f(x+h)-f(x)}{h} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

or rate of change

$$\underline{\frac{f(x+h)-f(x)}{h} = \text{average rate of change}}$$

$$\underline{f'(x) = \text{instantaneous rate of change}}$$

Panel 16

**Ex:** Suppose the position function of an object is given by $f(t) = 3t^2 + 5$ in miles

a) Find avg. rate of change over $[2,3]$ hours $= \frac{f(3)-f(2)}{3-2} = \frac{32-17}{1} = \underline{\underline{15}}$ $\underline{\underline{\text{mph}}}$

b) how fast is the car going when $t=2$? $f'(2)$

$f'(t) = 6t \Rightarrow f'(2) = \underline{\underline{12}}$, $f'(3) = \underline{\underline{18}}$

Panel 17

Ex: Let $p = 100 - q^2$ be a demand function.
How fast is price changing when $q = 5$?

Find rate of change at $q = 5$

$p'(5) = \underline{\underline{-10}}$

$p'(q) = -2q$

Panel 18

Ex: Suppose revenue function is $R(q) = 9q - q^3$
and the current level of production is at
$q = 2$ (thousand). Should you increase or
decrease production?

quiz on Wed

Find $R'(q) = 9 - 3q^2$

$R'(2) = 9 - 12 = \underline{\underline{-3}}$ , i.e. Revenue is decreasing

$\Rightarrow$ decrease production will get higher Revenue!

What production level gets me max. Revenue?

Next time