Panel 1

New Subject Entirely: Calculus (chapter 10 - 14)

Want to look at $f(x) = \frac{x^2-1}{x-1}$ can not plug in $x=1$

What about x close to 1?

x	$f(x)$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
⋮	⋮
	2

x	$f(x)$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001
	⋮
	2

1

Panel 2

Def of Limit: $\lim_{x \to a} f(x) = L$ (limit as x approaches a equals L)

means that as x gets closer and closer to a, f(x) gets closer to L.

Ex: $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

$\lim_{x \to 1} 2x+3 = 5$

$\lim_{x \to 0} \frac{x^2-4}{x-2} = 2$

$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = 4$

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Panel 3

Limits may or may not exist!

$\lim_{x \to 0} \frac{1}{x} = \pm\infty$ (undefined, not a number)

Note: $\frac{0}{\#} = 0$ $\lim_{x \to 0} \frac{x}{x-1} = 0$

$\frac{\#}{0} =$ undefined $\lim_{x \to 1} \frac{x}{x-1} =$ undefined

$\frac{0}{0} =$ more work $\lim_{x \to 0} \frac{x}{x^2 - x} = \lim_{x \to 0} \frac{\cancel{x}}{x(x-1)} = -1$

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Panel 4

How to find limits:

$\lim_{x \to a} f(x)$: what happens to $f(x)$ as x gets closer and closer to a, but is not equal to a.

First, you check - plug in $x = a$ anyway. If it worked ✓

or: $\frac{0}{\#} \to 0$ or $\frac{\#}{0} \to$ undf. or $\frac{0}{0} \to$ work

usually factoring

If all fails, make a table

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Panel 5

Find the following limits:

a) $\lim_{x\to 0} \frac{x-1}{x}$ undefined

b) $\lim_{x\to 1} \frac{x-1}{x} = 0$ kind of easy

c) $\lim_{x\to 0} \frac{x-1}{x^2-1} = 1$ easy

d) $\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{0}{0} = \lim_{x\to 1} \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{2}$

hard

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Panel 6

Today: $\lim_{x\to 0} \frac{e^{2x}-1}{x} = 2$ (guess)

pick 0.00001

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{(x+2)\cancel{(x-2)}} = \lim_{x\to 2} \frac{x-1}{x+2} = \frac{1}{4}$$

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Panel 7

One - Sided Limit

Say we have $f(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x+1 & \text{if } x \geq 0 \end{cases}$ Then, clearly

$$\lim_{x \to 2} f(x) = 5 \quad \text{and} \quad \lim_{x \to -2} f(x) = 12$$

But what is $\lim_{x \to 0} f(x) = ?$

Def: $\lim_{x \to a^+} f(x)$ means x close to a, but bigger than a (right-limit)

$\lim_{x \to a^-} f(x)$ means x close to a, but less than a (left-limit)

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Panel 8

Theorem: $\lim_{x \to a} f(x) = L$ if and only if

$\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Ex: $f(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x+1 & \text{if } x > 0 \end{cases}$ Find $\lim_{x \to 0} f(x) =$ undef

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x+1 = 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3x^2 = 0$

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Panel 9

$$f(x) = \begin{cases} 3x^2 - 1 & \text{if } x < 1 \\ 5x - 3 & \text{if } x > 1 \end{cases} \qquad \lim_{x \to 1} f(x) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5x - 3 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x^2 - 1 = 2$$

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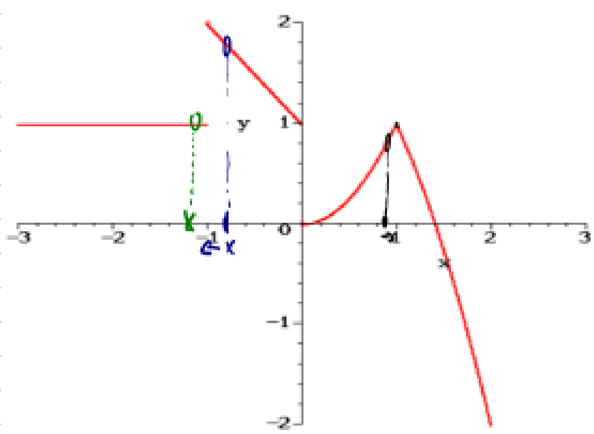
Panel 10

Graphical limits



$$\lim_{x \to -1^-} f(x) = 1$$

$$\lim_{x \to -1^+} f(x) = 2$$

$$\lim_{x \to 0^-} f(x) = 1$$

$$\lim_{x \to 0^+} f(x) = 0$$

$$\lim_{x \to 1^-} f(x) = 1$$

$$\lim_{x \to 1^+} f(x) = 1$$

① pick x close to a, on correct side

② slide x towards a and watch f(x)

↑ height

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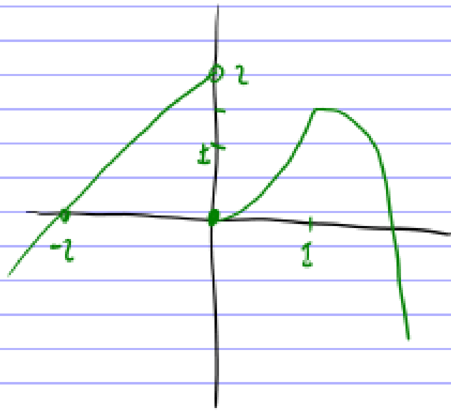
Panel 11



a) $\lim_{x \to -2^-} f(x) = 0$

b) $\lim_{x \to 0^+} f(x) = 0$

c) $\lim_{x \to 1^-} f(x) = 1.5$

d) $\lim_{x \to -2} f(x) = 0$

f) $\lim_{x \to 1} f(x) = 1.5$

e) $\lim_{x \to 0} f(x) =$ undef

Panel 12

Limits at Infinity

Def: $\lim_{x \to \infty} f(x)$ means: what does $f(x)$ do as x gets larger and larger and larger ......

Ex: $\lim_{x \to \infty} \frac{1}{x^2} = 0$

$\lim_{x \to -\infty} x^3 = -\infty$ (undef. as a number)

$\lim_{x \to -\infty} \frac{1}{x} = -0 = 0$

Panel 13

Limits at Infinity for Rational Functions

$$\lim_{x\to\infty} \frac{x^3 - 2x - 7}{3x^3+2} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(1 - \frac{2}{x^2}^{\,0} - \frac{7}{x^3}^{\,0}\right)}{\cancel{x^3}\left(3 + \frac{2}{x^3}^{\,0}\right)} = \underline{\underline{\frac{1}{3}}}$$

$$\lim_{x\to\infty} \frac{x^2 - 7 - x}{x^3 - 7x + 9} = 0$$

$$\lim_{x\to\infty} \frac{x^4 - 7}{x^3 - 8x^2 + 9x} = \text{undef.}$$

Rule: think of it as a race and consider the highest powers only!

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Panel 14

Limits at Infinity:

$$\lim_{x\to\pm\infty} \frac{p(x)}{q(x)} = \begin{cases} \deg(\text{top}) < \deg(\text{bottom}) : & 0 \\ \deg(\text{top}) = \deg(\text{bottom}) : & \#/\# \\ \deg(\text{top}) > \deg(\text{bottom}) : & \text{undef } (\pm\infty) \end{cases}$$

Ex: a) $\lim_{x\to-\infty} \frac{x^2 - 7x + 9}{3 - 4x^2} = -\frac{1}{4}$ $\boxed{\text{Quiz on Wed}}$

b) $\lim_{x\to\infty} \frac{3x^2 - 7}{9x^3 - 5} = 0$

c) $\lim_{x\to-\infty} \frac{3 - 4x + 5x^3}{7x^2 - 9} = \cancel{+\infty}$ $\underline{\text{undef}}$

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