Panel 1

Last Time:

$\lim_{x \to a} f(x) = L$ means: as x approaches a, $f(x)$ approaches L

$\lim_{x \to a^+} f(x)$ x approaches a, but $x > a$

$\lim_{x \to a^-} f(x)$ x approaches a, but $x < a$

$f(x)$ is continuous at $x = a$ if

(1) $f(a)$ exists

(2) $\lim_{x \to a} f(x)$ exists

(3) (1) = (2)

Graphically: graph has no holes or gaps

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Panel 2

Ex: Find the following limits:

$$\lim_{x \to 0} \frac{x^2 - 2x + 1}{x^2 - 4} = -1/4$$

$$\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 4} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+4)}{(x+2)\cancel{(x-2)}} = 6/4 = 3/2$$

$$\lim_{x \to 1^+} f(x) \text{ where } f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 1 \\ 3x & \text{if } x > 1 \end{cases}$$

$\lim_{x \to 1} f(x) =$ d.n.e. $\lim_{x \to 1^-} f(x) = 0$, $\lim_{x \to 1^+} f(x) = 3$

Is $f(x)$ continuous at $x = 1$? No

(1) $f(1)$ = none

(2) $\lim_{x \to 1} f(x)$ = d.n.e.

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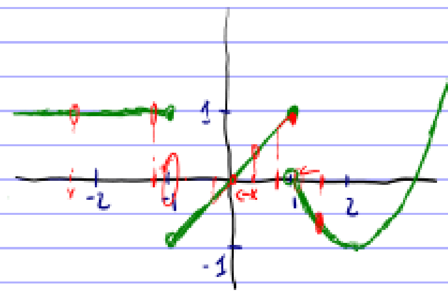
Panel 3

Ex: Consider the graph of the function shown below



a) $\lim_{x \to -2} f(x) = 1$

b) $\lim_{x \to 0^+} f(x) = 0$

c) $\lim_{x \to -1^-} f(x) = 1$

d) $\lim_{x \to 1^+} f(x) = 0$, $\lim_{x \to 1^-} f(x) = 1$

e) $\lim_{x \to 0} f(x) = 0$

f) $\boxed{\lim_{x \to -1} f(x) = \text{d.n.e.}}$

g) $\lim_{x \to 1} f(x) = \text{d.n.e.}$

h) Is f continuous at

$x = -1$? no

$x = 0$? yes

$x = 1$? no

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Panel 4

$f(x) = 5 + 2x$

later

$$\lim_{h \to 0} \boxed{\frac{f(x+h) - f(x)}{h}}$$

$$\lim_{h \to 0} \frac{(5 + 2(x+h)) - (5 + 2x)}{h} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} = 2$$

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Panel 5

Limits at Infinity

Def: $\lim_{x \to \infty} f(x)$ is that number that f(x) gets close to as x gets very large!

Ex: $\lim_{x \to \infty} \frac{1}{x^2} = 0$ "$\frac{1}{\infty} = 0$"

$\lim_{x \to -\infty} x^3 = -\infty$ (or d.n.e.)

$\lim_{x \to -\infty} \frac{1}{x} = -0 = 0$ "$\frac{1}{-\infty} = 0$"

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Panel 6

Limits at Infinity for Rational Functions

$$\lim_{x \to \infty} \frac{x^3 - 2x - 7}{3x^3 + 2} = \lim_{x \to \infty} \frac{\cancel{x^3}(1 - \frac{2}{x^2} - \frac{7}{x^3})}{\cancel{x^3}(3 + \frac{2}{x^3})} = \frac{1}{3}$$

factor highest power

$$\lim_{x \to \infty} \frac{x^2 - 7 - x}{x^3 - 7x + 9} = \lim_{x \to \infty} \frac{x^2(1 - \frac{7}{x^2} - \frac{1}{x})}{x^3(1 - \frac{7}{x^2} + \frac{9}{x^3})} = \lim_{x \to \infty} \frac{1}{x} \cdot 1 = 0$$

$$\lim_{x \to \infty} \frac{x^4 - 7}{x^3 - 8x^2 + 9x} = \frac{x^4(\ldots)}{x^3(\ldots)} = x(\ldots) \to \text{not a number!}$$

Rule:

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Panel 7

Rule for Limits at Infinity

$\lim_{x\to\infty} \frac{p(x)}{q(x)}$, p and q are polynomials

(1) deg (top) < deg (bottom) ⇒ limit is zero

(2) deg (top) = deg (bottom) ⇒ limit is # (highest coefficients)

(3) deg (top) > deg (bottom) ⇒ limit d.n.e.

Ex: A) $\lim_{x\to\infty} \frac{5-7x-9x^2}{3x^2+5} = \frac{-9}{3} = -3$

B) $\lim_{x\to-\infty} \frac{5x-7x^2}{2x^4-3x^2-1} = 0$

C) $\lim_{x\to\infty} \frac{5x^3-7x+9}{x^2+9x}$ = d.n.e.

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Panel 8

$\lim_{x\to\infty} \frac{5-5x-5x^2}{3x^3+3x-7x^2+1} = 0$

$\lim_{x\to\infty} \frac{5x^2-7x+9}{10x^2-3x} = \frac{5}{10} = \frac{1}{2}$

$\lim_{x\to\infty} \frac{5x-9x^4}{7x^8-4x^7} = 0$

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Panel 9

Applications of Limits

Ex: The population of a small city is $P(t) = 50{,}000 - \frac{2000}{t+1}$, t = years

a) What is the current population 48000

b) What is the population in the long run? $\lim_{t\to\infty} 50000 - \frac{2000}{t+1} = 50000$

Ex: For a host-parasite relation it was determined that when the host density is x (# of hosts per area) the number of host parasites is

$$y = \frac{900x}{10+45x} \qquad \lim_{x\to\infty} \frac{900x}{10+45x} = \frac{900}{45} = 20$$

If hosts increase without bound, what about parasites?

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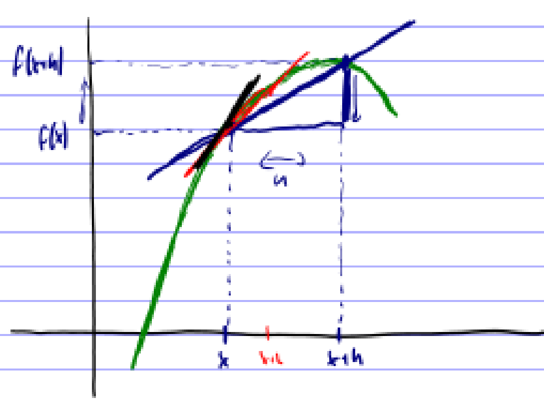
Panel 10

The Most Famous Limit of All:

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is slope of tangent line, i.e. that line that barely touch the graph



$\frac{f(x+h)-f(x)}{h} = \frac{\text{rise}}{\text{run}}$ is the slope of secant line through the 2 points

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Panel 11

The Derivative

If $f(x)$ is a function, we define the derivative of f as

$$\overset{\text{f prime}}{f'(x)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Geometrically, $f'(x)$ is the slope of the tangent line at x

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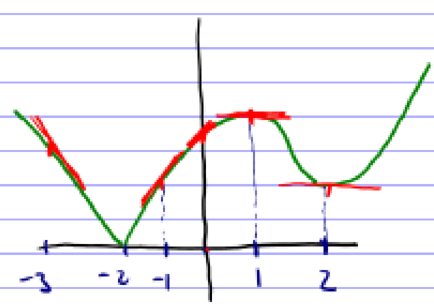
Panel 12

Ex: Find the indicated quantities: (signs only)



$f'(-3)$ negative

$f'(-1)$ positive

$f'(0)$ pos

$f'(1)$ zero

$f'(2)$ zero

$f'(1.5)$ negative

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Panel 13

Ex: Find the derivative to $f(x) = x^2$ at the point $x = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \qquad \boxed{f'(x) = 2x}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h = \underline{2x}$$

at $x = 1$, deriv. is $2 = f'(1)$

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Panel 14

Equation of tangent line to $f(x) = x^2 - 2$ at $x = 1$.

Want: $y - (-1) = 2(x - 1)$

slope of tangent at $x = 1$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - 2] - [x^2 - 2]}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2} - \cancel{x^2} + \cancel{2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x$$

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Panel 15

Derivatives as limits can get complicated ⇒ need short cut:

Ex: $f(x) = x \Rightarrow f'(x) =$

next time

$f(x) = x^2 \Rightarrow f'(x) =$

$f(x) = x^3 \Rightarrow f'(x) =$

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